ORDERS IN RINGS WITH INVOLUTION
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ORDERS IN RINGS WITH INVOLUTION

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ABSTRACT

In this paper we introduce a definition of order in a (not-necessarily unital) ring with involution in terms of the notions of Moore–Penrose inverse and ∗-cancellable element instead of those of group inverse and cancellable element. The main result states that if \( R \) is a Fountain–Gould order in a ring \( Q \) with \( Q \) semiprime and coinciding with its socle, then every involution \( ∗ : R \to R \) can be extended to a (unique) involution on \( Q \) in such a way that \( (R, ∗) \) is a ∗-order in \( (Q, ∗) \). And conversely, every ∗-order in an involution ring \( (Q, ∗) \) with \( Q \) semiprime and coinciding with its socle is a Fountain–Gould order in \( Q \).

Key Word: Quotient ring

In 1990 J. Fountain and V. Gould introduced a notion of left order in a ring which need not have a unity [1]. This notion involved that of group inverse. If the ring is endowed with an involution, it seems to be natural to consider the notion of Moore–Penrose inverse to define when a ring with involution is a ∗-order in another one.

**Definition 1 (See e.g. [2]).** Let \( R \) be a ring and \( a \) an element in \( R \). We say that \( b \in R \) is the **group inverse** of \( a \) if

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aba = a, bab = b and ab = ba.

Suppose that ∗: R → R is an involution. An element b ∈ R satisfying

\[ ab^*a = a, \quad ba^*b = b, \quad a^*b = b^*a \quad \text{and} \quad ba^* = ab^* \]

is said to be the Moore–Penrose inverse of a.

It is a well-known fact that the group inverse and the Moore–Penrose inverse, if they exist, are unique. Denote by \( a^\dagger \) and \( a^\# \) the group-inverse and the Moore–Penrose inverse, respectively, of an element a.

**Definition 2 (See [1]).** An element a in a ring R is left (right) square cancellable if for every \( x, y \in R^1 \),

\[ a^2 x = a^2 y \text{ implies } ax = ay \quad (xa^2 = ya^2 \text{ implies } xa = ya), \]

where \( R^1 = R \cup \{1\} \). A left and right square cancellable element is called square cancellable.

Suppose that ∗: R → R is an involution. Then a is said to be left (right) ∗-cancellable if

\[ aa^*x = aa^*y \text{ implies } a^*x = a^*y \quad (xa^*a = ya^*a \text{ implies } xa^* = ya^*) \]

for every \( x, y \in R^1 \). A left and right ∗-cancellable element is called ∗-cancellable.

Observe that being square cancellable (∗-cancellable) is a necessary condition for an element to have a group (Moore–Penrose) inverse. The set of all square cancellable elements of R will be denoted by \( S(R) \) while \( S^*(R) \) and \( S^r(R) \) will denote the sets of all ∗-cancellable elements and right ∗-cancellable elements, respectively, of R.

**Definition 3 (See [1]).** We define a subring R of a ring Q to be a Fountain–Gould left order in Q when the following conditions hold:

1. Each square cancellable element of R has a group inverse in Q.
2. Each element \( q \in Q \) can be written as \( q = a^\dagger b \), with a in \( S(R) \) and b in R.

We also say that Q is a Fountain–Gould ring of left quotients of R. Similarly we define Fountain–Gould right order and Fountain–Gould ring of right quotients.

If R is both a left and a right Fountain–Gould order in Q then we say that R is a Fountain–Gould order in Q and that Q is a Fountain–Gould ring of quotients of R. When condition (2) alone is satisfied, R is called a weak order in Q.

**Definition 4.** A subring \((R, \ast)\) of an associative ring with involution \((Q, \ast)\) is said to be a ∗-order in \((Q, \ast)\) whenever:
ORDERS IN RINGS WITH INVOLUTION

(1) Each ∗-cancellable element of \( R \) has a Moore–Penrose inverse in \( Q \).
(2) Each element \( q \) of \( Q \) can be written in the form \( q = a^\sharp b \), where \( a \in S^\star(R) \) and \( b \in R \).

When condition (2) alone is satisfied, then \( R \) is called a weak ∗-order in \( Q \).

Observe that if \((R, ∗)\) is a ∗-order in \((Q, ∗)\), given \( q \in Q \) there exist \( a \in S^\star(R) \) and \( b \in R \) such that \( q^\star = a^\sharp b \), hence \( q = b^\star a^\sharp = b^\star a^\star \) with \( a^\star \in S^\star(R) \), so it is not necessary to distinguish between left and right ∗-orders.

The notion of ∗-order extends that of classical order in an involution ring since every ∗-order in a unital ring with involution \( Q \) is a classical order in \( Q \) (Corollary 8).

From now on we will use the following lemma without mentioning it.

**Lemma 5.** If \((R, ∗)\) is a ∗-order in an involution ring \((Q, ∗)\) and \( q \in Q \), we can write

\[
\begin{align*}
(i) & \quad q = a^\sharp b, \quad \text{with } a^\star a^\sharp b = b, \\
(ii) & \quad q = c^\sharp d, \quad \text{with } c^\star c^\sharp d = d, \\
(iii) & \quad q = xy^\sharp, \quad \text{with } x^\star y^\sharp x = x, \\
(iv) & \quad q = zt^\sharp, \quad \text{with } zt^\star t^\sharp = z,
\end{align*}
\]

for some \( b, d, x, z \in R \), \( a, c, y, t \in S^\star(R) \).

**Proof:** If \( q \in Q \), then \( q = x^\sharp y \), with \( x \in S^\star(R) \), \( y \in R \).

(i) Notice that \( q = x^\sharp x^\star x^\sharp x^\star x^\sharp y \). Take \( a = xx^\star x \), \( b = x^\star xy \). Then \( a \in S^\star(R) \), \( a^\star = x^\star x^\star x^\sharp \), \( b \in R \), and \( q = a^\star b \), with \( a^\star a^\sharp b = b \).

(ii) Since \( q = x^\sharp x^\star x^\sharp \), if we take \( c = xx^\star, d = xy \), then \( c \in S^\star(R) \) with \( c^\star = x^\star x^\star \), \( d \in R \), and \( q = c^\sharp d \) with \( c^\star c^\sharp d = d \).

For (iii) and (iv) we only need to consider the element \( q^\star \) and the previous items.

**Lemma 6.** If \((R, ∗)\) is a weak ∗-order in an involution ring \((Q, ∗)\) then \( R \) is a Fountain–Gould weak order in \( Q \).

**Proof:** Given \( q \in Q \) let \( a, b \) be in \( S^\star(R) \) and \( R \), respectively, such that \( q = a^\sharp b \) with \( a^\star a^\sharp b = b \). Denote \( x = aa^\star \) and \( y = ab \). It is not difficult to see that \( x \in S(R) \) with \( x^\dagger = a^\sharp (a^\star)^\dagger \) and \( q = x^\dagger y \).

**Theorem 7** (Common denominator property). Suppose that \((R, ∗)\) is a ∗-order in an involution ring \((Q, ∗)\). Then, given \( q_1, \ldots, q_n \in Q \) there exist \( c \in S^\star(R) \), \( d_1, \ldots, d_n \in R \) such that \( c^\star c^\sharp d_i = d_i \) and \( q_i = c^\sharp d_i \).

**Proof:** By Lemma 6, \( R \) is a Fountain–Gould weak order in \( Q \) and by [3, Theorem 5], which is valid for Fountain–Gould weak orders, there exist \( a \in S(R) \),
\[ b_i \in R \text{ such that } q_i = a^\dagger b_i. \] Write \[ a^\dagger = c^\ast b \] with \[ c^\ast c^\dagger b = b. \] Then \[ q_i = c^\ast (bb_i) \] and \[ c^\ast c^\dagger bb_i = bb_i. \]

**Corollary 8.** Suppose that \((R, \ast)\) is a \(\ast\)-order in a unital ring \((Q, \ast)\). Then \(R\) is a classical order in \(Q\).

**Proof:** Take a regular element \(a\) of \(R\). Since \(a\) is, clearly, in \(S(R)\), there exists \(a^\ast \in Q\) satisfying \[ aa^\ast a = a. \] Apply that \(a\) is in \(S(Q)\) too to get \[ aa^\ast = a^\ast a = 1. \] Now, take \(q \in Q\). By the common denominator property, there exist \(x \in S(R^*)\), \(y, z \in R\) such that \[ q = x^\ast y \text{ with } x^\ast x^\dagger y = y \text{ and } 1 = x^\ast z \text{ with } x^\ast x^\dagger z = z, \] so \(z = x^\dagger\) and \(x^\dagger = (x^\ast)^{-1}. \)

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**THE LOCAL RINGS AT AN ELEMENT**

The following definition was introduced by the author jointly with Fernández López, García Rus and Gómez Lozano in [4]. The use of the local rings at elements of a ring \(R\) allows to exchange information among them (the ring and its local rings at elements). Moreover, Fountain–Gould orders can be characterized by using these local rings at elements.

**Definition 9.** Let \(R\) be a ring and \(x \in R\). Then the subset \(xRx\) equipped with the multiplication defined by

\[ (xr)^\ast = xxrxx \]

is a ring called the **local ring of** \(R\) **at** \(x\) and denoted by \(Rx\).

Observe that if \(e\) is an idempotent of \(R\) then the local ring of \(R\) at \(e\), \(Re\), coincides with the subring \(eRe\) of \(R\) and if \(x \in R\) is von Neumann regular, then \(Rx\) is unital with \(x\) as the unity.

Local algebras at elements, which were introduced by K. Meyberg (see [5]), are usually presented in a different way: let \(R\) be an algebra and \(x \in R\). Recall that \(R\) endowed with the \(x\)-**homotope product** \[ a \cdot_x b = axb \] becomes an associative algebra, the \(x\)-**homotope**, \(R^x\), which has as an ideal the set \(\text{Ker}(x) := \{a \in R \mid xa = 0\}\). Moreover, the mapping \(a + \text{Ker}(x) \mapsto xa\) is an isomorphism from \(R^x/\text{Ker}(x)\) onto our local algebra \(Rx\). However our definition is more suitable for our purposes because for every element \(x \in R\), \(Rx\) is contained in \(R\).

**Proposition 10** (See [4, Proposition 8.3]). Let \(R\) be a subring of a simple ring \(Q\) which coincides with its socle. Then \(R\) is a Fountain–Gould order in \(Q\) if and only if for every \(q \in Q\) there exists \(x \in R\) satisfying:

- \(x\) is von Neumann regular in \(Q\),
- \(q \in xQx\) and
- \(Rx\) is a classical order in the unital ring \(Qx\).
EXTENDING THE INVOLUTION

In [6, Proposition 2.3] the author proves that if \( R \) is a classical order in a unital ring \( Q \) then every involution on \( R \) can be extended to a (unique) involution on \( Q \). We will prove that every involution of a Fountain–Gould order can be extended to the ring of quotients when this is semiprime and coincides with its socle.

**Definition 11.** Let \( Q \) be a ring. We say that \( Q \) satisfies \( M_L \) (or \( M_R \)) if for each \( a \in Q \) the following sequence is stationary (see [1]):

\[
a Q^1 \supseteq a^2 Q^1 \supseteq a^3 Q^1 \supseteq \cdots
\]

(or \( Q^1 a \supseteq Q^1 a^2 \supseteq Q^1 a^3 \supseteq \cdots \)).

If \( * : Q \to Q \) is an involution we say that \((Q, *)\) satisfies \( M^*_L \) (or \( M^*_R \)) if for every \( a \in Q \) the following sequence is stationary:

\[
a Q^1 \supseteq a^* a Q^1 \supseteq a^* a^* a Q^1 \supseteq \cdots
\]

(or \( Q^1 a \supseteq Q^1 a^* a \supseteq Q^1 a^* a^* a \supseteq \cdots \)).

Notice that since \( * \) is an involution, \((Q, *)\) satisfies \( M^*_L \) if and only if it satisfies \( M^*_R \). If for an element \( a \in Q \) the following sequence stabilizes:

\[
a Q^1 a \supseteq a^* a Q^1 a^* a \supseteq a^* a^* a Q^1 a^* a \supseteq \cdots
\]

we say that the element \( a \) is **strongly \( \pi \)-\( * \)-regular**. If every element in \( Q \) is strongly \( \pi \)-\( * \)-regular, we say that \((Q, *)\) is **strongly \( \pi \)-\( * \)-regular**.

The relation among the defined chain conditions is studied in the following lemma.

**Lemma 12.** Let \((Q, *)\) be an associative ring with involution and consider the following conditions:

(i) \( Q \) satisfies \( M_L \).

(ii) \((Q, *)\) satisfies \( M^*_L \).

(iii) \((Q, *)\) is strongly \( \pi \)-\( * \)-regular.

Then (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii).

**Proof:** (i) \( \Rightarrow \) (ii). By hypothesis, the sequence

\[
a^* a Q^1 \supseteq (a^* a)^2 Q^1 \supseteq (a^* a)^3 Q^1 \supseteq \cdots
\]

terminates. Let \( m \) be such that \((a^* a)^m Q^1 = (a^* a)^{m+1} Q^1 \). Then

\[
a^* \ldots a^* a Q^1 \supseteq a^* \ldots a^* a Q^1 \supseteq a^* \ldots a^* a Q^1.
\]

\[
\text{ for } m, m+1, m+2.
\]
As the first and the third term coincide, containings must be equalities, so (ii) is satisfied.

(ii) ⇒ (iii). As \((Q, \ast)\) satisfies \(M_1^*\) and \(M_2^*\), there exist \(p, q \in \mathbb{N}\) such that \((aa^*)^p Q^1 = (aa^*)^{p+1} Q^1\) and \(Q^1(a^*a)^q = Q^1(a^*a)^{q+1}\). If we take \(m = \max\{p, q\}\), then

\[a Q^1 a \supseteq aa^* Q^1 a^* a \supseteq aa^* a Q^1 a a^* a \supseteq \cdots\]

stabilizes at \(m\).

\(\square\)

**Proposition 13.** Let \((Q, \ast)\) be an involution ring with \(Q\) satisfying \(M_1\). Then every \(*\)-cancellable element of \(Q\) has a Moore–Penrose inverse in \(Q\).

**Proof:** Take \(a \in S^*(Q)\). By Lemma 12 the sequence

\[a Q^1 a \supseteq aa^* Q^1 a^* a \supseteq aa^* a Q^1 a a^* a \supseteq \cdots\]

is stationary. Let \(n\) be a natural number such that

\[(aa^*)^n Q^1(a^*a)^n = (aa^*)^{n+1} Q^1(a^*a)^{n+1} = \cdots\]

In particular there exists \(q \in Q^1\) such that \((aa^*)^p(a^*a)^q = (aa^*)^{p+2}(a^*a)^{q+2}\). Since \(a\) is \(*\)-cancellable, and consequently \(a^*\) is \(*\)-cancellable too, it follows \(a = aa^* a q a^* a a^* a \in aa^* Q a^* a\). By [7, Theorem 1], \(a\) has a Moore–Penrose inverse in \(Q\).

\(\square\)

**Theorem 14.** Let \((R, \ast)\) be a weak \(*\)-order in an involution ring \((Q, \ast)\). Then:

(i) \(S^*(R) \subseteq S^*(Q)\).

(ii) \((R, \ast)\) is a \(*\)-order in \((Q, \ast)\).

(iii) \(R\) is a Fountain–Gould order in \(Q\).

**Proof:**

(i) Let \(a\) be in \(S^*(R)\) and suppose \(pa^*a = qa^*a\). If \(p = q = 1\) the result is trivial. If \(p = 1\) and \(q \in Q\), by the common denominator property there exist \(u \in S^*(R), v, w \in R\) satisfying \(a^* = u^v v\) and \(q = u^w w\). Then \(a^*a = qa^*a\) implies \(a^*a = u^v w a^* a\). Multiply by \(uu^*\) on the left.

We have \(uu^* a^* a = uu^* a^* a\). Since \(a \in S^*(R), uu^* a^* = uu^* a^* a\) and hence \(a^* = qa^*\). Finally, consider \(p, q \in Q\) and let \(u \in S^*(R)\) and \(v, w \in R\) such that \(p = u^v v\) and \(q = u^w w\). Then \(pa^*a = qa^*a\) implies \(u^v v a^* a = u^v w a^* a\) and multiplying by \(uu^*\) on the left hand side we obtain \(uu^* a^* a = uu^* a^* a\). Now \(a \in S^*(R)\) implies \(u a^* = uu^* a^*\), which gives, multiplying by \(u^v u^w\) on the left hand side, \(pa^* = qa^*\).

(ii) Take \(a \in S^*(R)\). Then \(a^* \in S^*(R)\) and by (i), \(a, a^* \in S^*(Q)\), that is, \(a \in S^*(Q)\). Apply Proposition 13 to prove that \(a\) has a Moore–Penrose inverse in \(Q\).
Theorem 15. Suppose that \( R \) is a Fountain–Gould order in a simple ring \( Q \) which coincides with its socle. If \( * : R \to R \) is an involution, then there is a unique involution \( * \) which extends that of \( R \). Moreover \((R, *)\) is a \(*\)-order in \((Q, *)\).

Proof: We define

\[
\circ : Q \to Q
\]

\[
a^\circ b \mapsto b^*a^\circ
\]

which makes sense because \( a \in S(R) \) implies \( a^* \in S(R) \). The map \( \circ \) is well defined as we can see from [1, Lemma 5.3 and Lemma 5.4]. Taking into account the common denominator property for Fountain–Gould orders and [1, Lemma 5.4], we prove that \( \circ \) is an involution. It is clear that \( \circ \) extends \(*\), and to prove uniqueness, suppose that \( \circ : Q \to Q \) is another involution which extends \(*\). Then \((a^\circ b)^\circ = b^\circ a^\circ = (a^\circ b)^\circ \), so \( \circ = \circ \) (notice that \( a^\circ = a^\circ \), as it is easy to see from Definition 1).

Finally, by Theorem 14 (ii), it is enough to prove that \((R, \circ)\) is a weak \(*\)-order in \((Q, \circ)\). Take \( q \in Q \). By the *-Litoff Theorem [8, Theorem 4.6.15] there exists a symmetric idempotent \( e_\circ \) in \( Q \) such that \( q \in Q_\circ \) and \( Q_\circ \) is isomorphic to a \( n \times n \) matrix ring over a division ring with involution \( D \). Moreover, there exist minimal orthogonal idempotents \( e_1, \ldots, e_n \) in \( Q_\circ \) (and consequently in \( Q \)) such that \( e_i = e_i^\circ \) for every \( i \), \( e = \sum_{i=1}^n e_i \) with and \( m = n \) or \( e_1 \perp e_i^\circ \) for every \( i \), \( e = \sum_{i=1}^n (e_i + e_i^\circ) \), \( m = 2n \) and \( D \) is a field (with the identity as involution).

In the first case, \( e_i = e_i^\circ \) for every \( i \in \{1, \ldots, n\} \), take \( u \in \{e_1, \ldots, e_n\} \). As \( R \) is a Fountain–Gould order in \( Q \), by Proposition 10, there exists \( x \in R \) such that \( u \in Q_\circ \) and \( R_\circ \) is a classical order in \( Q_\circ \), so there exists a regular element \( xru \) in \( R_\circ \) which is a left denominator for \( u \), that is, \( 0 \neq xru \in R_\circ \).

Since \( u \) is a minimal idempotent, \( Qxru = Qu \) and \( u = u^\circ \in (Qu)^\circ = u^\circ r^\circ x^\circ Q = ur^\circ x^\circ Qxru \). But by primeness of \( R \), \( 0 \neq ur^\circ x^\circ Qxru \subseteq R \cap Q_\circ \), and since \( Q_\circ \) is a division ring, every nonzero element from \( ur^\circ x^\circ Qxru \), say \( a \), for example, is invertible in \( Q_\circ \), hence \( a \in S^*(R) \) and \( u \in Qua^* = aa^*Qua^* = Qa \).

With this reasoning we have found an element \( b \in R \), \( b = a_1a_1^\circ + \cdots + a_na_n^\circ \), with \( a_i \in S^*(R) \cap Q_\circ \) for \( i \in \{1, \ldots, n\} \) such that \( Q_\circ = Qb \). As \( R_\circ \) is a classical order in \( Q_\circ \), there exist \( bab \in \text{Reg}(R_\circ) \), \( b^\circ b \in R_\circ \), such that \( b^\circ q = btb \cdot b^\circ b = btb^\circ b \), where \( btb \in Q_\circ \) denotes the inverse of \( bab \) in \( Q_\circ \), i.e., \( babtb = btbab = b \). Taking into account the fact that \( b^\circ = b^\circ \) (since \( b = b^\circ \) and \( b \in S(R) \cap S^*(R) \)), an easy computation shows, first that \( bab = ba^*b \) and \( btb = bt^*b \), and secondly that \( b^\circ t^\circ b^\circ = (b^\circ ab^\circ)^\circ \). Hence \( b^\circ q = btb^\circ b = b^\circ b^\circ t^\circ b^\circ b^\circ b^\circ = b^\circ (b^\circ ab^\circ)^\circ b^\circ b^\circ \), which implies (multiplying by \( b^\circ b \) on the left hand side) \( q = (b^\circ ab^\circ)^\circ b^\circ b \).
Suppose $e_i \perp e_i$ for every $i \in \{1, \ldots, n\}$.

$Q_e = De_i$, where $D$ is a field, and if $\alpha e_i \in R_u \setminus \{0\}$, $Q_{e_i} = Q_{\alpha e_i}$ and $Q_{e_i + e_i} = Q_{\alpha e_i + e_i}$.

Again we have found an element $b \in R$ such that $b = b^*$ and $Q_e = Q_b$. □

**UNICITY**

We prove, as a corollary of a more general result, that if $(R, *)$ is a $*$-order in a simple involution ring $(Q, *)$ with $Q$ coinciding with its socle, then this ring of $*$-quotients is unique up to $*$-isomorphisms.

**Definition 16.** If $(A, *)$ and $(B, *)$ are associative involutions and $\psi : A \to B$ a homomorphism of associative rings, we will say that $\psi$ is a $*$-homomorphism if for every $a \in A$, $\psi(a^*) = (\psi(a))^*$.

**Theorem 17.** Suppose $(R, *)$ is a $*$-order in a simple involution ring $(Q, *)$ which coincides with its socle; $(T, \otimes)$ an involution ring and $\psi : (R, *) \to (T, \otimes)$ a $*$-monomorphism such that $(\psi(R), *)$ is a $*$-order in $(T, \otimes)$. Then there exists a unique $*$-homomorphism $\theta : (Q, *) \to (T, \otimes)$ that extends $\psi$.

**Proof:** By Theorem 14 (iii) and [9, Proposition 3.6], $T$ is a simple ring which coincides with its socle. Apply [1, Theorem 5.5] to get that $R$ and $\psi(R)$ are Fountain–Gould orders in $Q$ and $T$, respectively. By [1, Corollary 5.6], there exists a unique isomorphism $\theta : Q \to T$ which extends $\psi$, so the only thing we have to prove is that $\theta$ is a $*$-isomorphism.

If $q = a^*b$ is an element in $Q$ with $a \in S^* (R)$ and $b \in R$, then $q = a^*a^*ab = (aa^*)^2ab$ and $(aa^*)^2 = (aa^*)^1$, so we can choose $c = c^* \in S^* (R) \cap S(R)$, $d \in R$ such that $q = c^*d^*c^1$.

Now $\theta(q^*) = \theta(d^*c^1) = \psi(d^*)\psi(c^1) = \psi(d)^\#\psi(c)^1 = (\psi(c)^\#\psi(d))^\# = \theta(c^1d^\#)^\# = \theta(q^\#)$. □

**Corollary 18.** If $(R, *)$ is a $*$-order in two involution rings $(Q, *)$ and $(T, \otimes)$, with $Q$ and $T$ simple and coinciding with their socles, then $Q$ and $T$ are $*$-isomorphic via a $*$-isomorphism that restricted to $R$ is the identity.

**THE MAIN THEOREM**

**Definition 19.** If $(R, *)$ is a ring with involution, by a $*$-ideal of $(R, *)$ we mean an ideal $I$ of $R$ such that $y^* \in I$ for every $y \in I$.

**Lemma 20.** Let $(R, *)$ be a $*$-order in a ring with involution $(Q, *)$, $Q$ simple and coinciding with its socle. If $I$ is a $*$-ideal of $R$, essential as an ideal of $R$, then $(I, *)$ is a $*$-order in $(Q, *)$. 

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ORDERS IN RINGS WITH INVOLUTION

Proof: Given \( q \in Q \), by Theorem 14 (iii) and Proposition 10, there exists an element \( u \in R \) such that \( q \in uQu \) and \( R_u \) is a classical order in the unital ring \( Q_u \). Observe that \( I_u \) defined as \( uIu \) with product \( uyu \cdot zu = uyuzu \) is an ideal of the local ring of \( R \) at \( u \). Moreover it is easy to see that \( I_u \) is an essential ideal of \( R_u \); if \( J \) is a nonzero ideal of \( R_u \), take \( 0 \neq uru \in J \). By Theorem 14 (iii) and [9, Proposition 3.6] \( R \) is prime hence \( Ruru \) is a nonzero ideal of \( R \). Essentiality of \( I \) in \( R \) allows us to find \( 0 \neq y \in Ruru \cap I \) and by primeness of \( R \) again, \( 0 \neq uRyRu \subseteq uRRuruRRu \subseteq I_u \cap J \).

Then, by [4, Proposition 7.5], \( I_u \) is also a classical order in \( Q_u \). Take a denominator \( uru \in Inv(Q_u) \cap I_u \) for \( q \). Then \( (I_u)_uru \) is a classical order in \( Q_uru \), that is, \( I_uru \) is a classical order in \( Q_uru \) and we have proved that given \( q \in Q \) there exists \( uru \in I \) (von Neumann regular in \( Q \)) such that \( q \in uruQuru \) and \( I_uru \) is a classical order in \( Q_uru \), hence the required result follows from Proposition 10 and Theorem 15.

Lemma 21. If \((Q, \ast)\) and \((Q', \ast')\) are associative rings with involution, \( Q' \) satisfies the descending chain condition on principal left ideals, \( \varphi : Q \to Q' \) is a \(*\)-epimorphism and \((R, \ast')\) is a \(*\)-order in \((Q, \ast)\), then \((\varphi(R), \ast')\) is a \(*\)-order in \((Q', \ast')\).

Proof: By Theorem 14 (ii) it is enough to prove that \((\varphi(R), \ast')\) is a weak \(*\)-order in \((Q', \ast')\). Take \( \varphi(q) \in Q' \). Since \( q \in Q \), there exist \( a \in S^*(R) \), \( b \in R \) such that \( q = a^b \). Then \( \varphi(q) = \varphi(a^b) = \varphi(a)^\varphi(b) = \varphi(a)^{\varphi(b)} \).

Proposition 22. If \((R, \ast)\) is a \(*\)-order in a semiprime ring \((Q, \ast)\) which coincides with its socle, \( Q = \oplus Q_i \) with \( Q_i \) simple \(*\)-ideals of \( Q \), and we denote by \( R_i \) the \(*\)-ideal (of \( R \)) \( R_i = R \cap Q_i \), then \((R_i, \ast)\) is a \(*\)-order in \((Q_i, \ast)\).

Proof: Let \( \pi_i \) be the projection map from \( Q \) to \( Q_i \). By the previous lemma, \((\pi_i(R), \ast)\) is a \(*\)-order in \((Q_i, \ast)\). We are going to prove that \( R \cap Q_i \) is an essential \(*\)-ideal of \( \pi_i(R) \), hence the result follows from Lemma 20.

If \( J \) is a nonzero ideal of \( \pi_i(R) \), take \( 0 \neq y \in J \). If \( y = a^b \) with \( a \in S^*(R) \), \( b \in R \) and \( a^a b = b \), then \( 0 \neq a^b y = b \in (R \cap Q_i) \cap J \).

The Main Theorem. Suppose that \( R \) is a subring of a semiprime ring \( Q \) which coincides with its socle.

(i) If \( R \) is a Fountain–Gould order in \( Q \) and \( \ast : R \to R \) is an involution, then there exists an involution over \( Q \) which extends that of \( R \) and \((R, \ast)\) is a \(*\)-order in \((Q, \ast)\).

(ii) If there is an involution \( \ast : Q \to Q \) such that \((R, \ast)\) is a \(*\)-order in \((Q, \ast)\), then \( R \) is a Fountain–Gould order in \( Q \).

Moreover, the ring \( Q \) is uniquely determined up to \(*\)-isomorphisms.
Proof: If \( Q \) were simple, the statement would follow from Theorem 15 and Theorem 14 (iii). In general, since \( \oplus R_i \subseteq R \subseteq \oplus Q_i = Q \). Proposition 22 and its analogue for Fountain–Gould orders allow us to reduce the problem to the simple case.

The unicity up to \(*\)-isomorphisms is Corollary 18. \( \square \)

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