Oscillatory dynamics of inviscid planar liquid sheets

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Abstract

One-dimensional models of inviscid, planar liquid sheets surrounded by dynamically passive gases are obtained by means of perturbation methods, integral formulations, series expansions and variational principles. It is shown that integral formulations, series expansions and variational principles provide analogous one-dimensional models under certain approximations for the velocity and pressure fields. The equations for slender and thin sheets obtained from integral formulations are shown to be asymptotically equivalent to the exact equations of inviscid, planar liquid membranes. Algebraic and differential methods are employed to determine the singularities of the steady equations obtained from integral formulations, and indicate that the liquid does not leave the nozzle with an angle equal to that of the exit if the Weber number is equal to or less than one. Numerical studies of the time-dependent governing equations are presented in order to illustrate the nonlinear dynamics and bifurcations of confined, inviscid, planar liquid sheets when they are subjected to time-dependent pressure differences or gases are injected on either side of the sheet, in the absence of heat and mass transfer between the gases and the liquid.

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1. Introduction

Planar liquid sheets have a variety of applications in coating processes, protection systems, shielding, sprays, etc. For example, they have been employed to apply protective organic coatings at high speed to continuous steel by pumping the coating material from a supply tank to a narrow slot where the liquid forms a sheet or curtain which falls on the material to be protected [1]. Planar liquid sheets play an important role in film casting or spinning processes at low Reynolds numbers [2,3]. Compressible planar jets are found in rocket, scramjet, ramjet, turbojet, and turbofan engines, thermal and plasma spraying devices, and astrophysics [4]. Planar liquid sheets or curtains have also been proposed to passively contain reacting gases within the cavity of a chemical laser [5]. If gravitational and viscous effects are neglected, planar liquid sheets may be used for determining the dynamic surface tension of liquids from the mass flow rate, the longitudinal velocity component and an angle [6].

The disintegration of planar liquid sheets with a coflow of high speed air due to Kelvin–Helmholtz and capillary instabilities plays a paramount role in determining the formation of ligaments and droplets because both instabilities are present in the axisymmetric air-blast atomizers which are employed in reciprocating engines and gas turbine combustors [7].

Two-dimensional potential flow models of unconfined, inviscid, planar liquid sheets rising or falling under gravity have been studied by several authors. Vanden–Broeck and Keller [9] formulated an integro-differential equation for two-dimensional flows to determine the two free surfaces of a planar liquid sheet and showed that the shape of the sheet only depends on the Froude number.

Small transverse displacements of vertically falling planar liquid sheets may be amplified by the Kelvin–Helmholtz instability, and, together with the feedback in a waterfall fountain, may result in periodic oscillations due to the compression and expansion of the gases that surround the liquid sheet. Casperson [11,12] used the two-dimensional Euler equations, a sinusoidal pressure perturbation at the gas–liquid interface together with a polytropic expression for the compression and expansion of the gases that surround the liquid sheet to determine the dynamic behaviour of fluttering fountains. The periodic oscillations observed by Casperson [11,12] are similar to the ones reported by the present writer in his studies of the nonlinear dynamics of inviscid annular liquid jets which form enclosed volumes; the compression and expansion of the gases enclosed in these volumes can result in relaxation oscillations [13,14].

The one-dimensional models which can be used to study the fluid dynamics of inviscid, planar liquids sheets or curtains can be classified into the following groups: models based on potential flow theory, approximate integral models, Cosserat-type models, models based on Taylor series expansions, and slender
models. This classification follows closely that for axisymmetric, inviscid jets and annular jets [8]. Keller and Weitz [10] proposed a system of four one-dimensional partial differential equations for the sheet’s thickness, vertical displacement and longitudinal and transverse velocity components of thin planar liquid sheets falling under gravity and subject to surface tension. According to the simple hydraulic theory of thin jets, each particle of a jet moves independently along a parabolic trajectory, and Keller and Weitz [10] considered how their hydraulic model is modified by surface tension.

A one-dimensional model of steady, inviscid planar liquid curtains was proposed by Finnicum et al. [15] who analyzed the effects of applied pressure on the shape of a liquid curtain falling under gravity. Their model is based on the one developed by Boussinesq and assumes that the velocity is constant across the liquid sheet and that the surrounding gases are dynamically passive; a similar model was developed previously by the author for inviscid, annular liquid jets [16]. Finnicum et al. [15] showed by means of Taylor series expansions that a singularity may exist at the nozzle exit or downstream from it if the Weber number is equal to or smaller than one, respectively, and indicated that this singularity is removable and that the liquid does not leave the nozzle with an angle equal to that of the nozzle exit for these Weber numbers. Ramos [17] showed that his generalization of the Boussinesq model for annular liquid jets [16] also exhibits removable singularities for Weber numbers less than or equal to one and determined these singularities by means of algebraic methods and Taylor’s series expansions.

One-dimensional models of unconfined, inviscid, planar liquid sheets have been derived by means of perturbation methods based on the slenderness ratio, i.e., the ratio of the sheet’s thickness at the nozzle exit to a characteristic longitudinal dimension or wavelength, by Weinstein et al. [18] and Clarke et al. [19] for Weber numbers of the order of unity and inviscid, slender liquid sheets subject to pressure differences and gravity, and surrounded by dynamically passive gases. These authors had to approximate the base flow solution so as to have a correct linearization for their linear stability analysis, and obtained some partial differential equations for the varicose and sinuous modes which were found to be uncoupled. Weinstein et al. [22] used potential flow theory and perturbation methods for slender, thin planar liquid sheets subject to gravity and pressure differences and found standing waves in agreement with those studied by Lin and Roberts [23] even though these authors observed the waves in viscous liquid curtains; a similar result was found by the author for inviscid planar liquid sheets and membranes [20,21]. De Luca and Costa [24] analyzed vertically falling, inviscid, thin liquid sheets based on the assumptions that the flow is potential and the gases surrounding the liquid are dynamically passive, employed the WKBJ approximation and derived the dispersion relations for the varicose and sinuous modes, which are in accord with those derived by Taylor [1] for small wave numbers.
One-dimensional model of (two-dimensional) inviscid, planar liquid sheets can also be developed by employing Taylor’s series expansions and second-degree polynomials for the pressure and the longitudinal and transverse velocity components under the assumption that the surrounding gases are dynamically passive. These models can also be obtained by integration of the two-dimensional Euler equations along the sheet thickness and by approximating the velocity components and pressure in the resulting integrals so that one can obtain a closed system of equations [8]. Such an integral approximation was formulated by this writer [25] in his derivation of a model for inviscid, annular liquid jets where, in addition, the pressure and surface tension terms were expanded about the midline of the jet.

The objective of this paper is sixfold. First, one-dimensional models based on potential flow theory and perturbation methods for slender nearly vertical, and slender and thin, confined, inviscid, irrotational, planar liquid sheets are derived and two flow regimes depending on the Weber number are identified.

Second, one-dimensional models for inviscid, planar liquid sheets are derived by means of an integral formulation based on the integration of the two-dimensional Euler equations across the sheet in an analogous manner to that employed by the author for annular liquid jets [25], variational formulations, and Taylor series expansions. It is shown that models based on the integration of the two-dimensional Euler equations and Taylor series expansions are identical if the same polynomials are employed for the pressure and the velocity components; these methods are, however, approximate ones and their accuracy is on the order of a power of the sheet’s thickness, and this power coincides with that of the first term of the Taylor series expansion that is neglected. Variational formulations are approximate techniques which satisfy the governing equations in a weak sense and involve moments of the pressure. For some velocity and pressure approximations, variational techniques provide the same one-dimensional equations as those derived from Taylor series expansions [8].

The third objective of this paper is to show that the equations for thin, planar liquid sheets obtained by means of an integral formulation are asymptotically equivalent to those for inviscid, planar liquid membranes, i.e., liquid sheets of zero thickness [21]. The fourth objective is to obtain approximate analytical solutions to the steady state equations derived from integral formulations.

The fifth objective is to determine the steady state singularities of the model for slender and thin sheets obtained from the integration of the Euler equations, by means of algebraic and differential techniques. Although the singularities of planar liquid membranes can be analyzed in a similar manner, they are not reported here [21]. The sixth and final objective of the paper is to determine numerically the nonlinear dynamics of confined planar liquid sheets (cf. Fig. 1) subject to a variety of time-dependent forcings.
2. Governing equations

Consider an inviscid liquid sheet infinitely long in the \( z \)-direction falling under gravity and emerging from the slot \((x = 0, -h_0/2 \leq y \leq h_0/2, z)\) with velocity \((u_0, v_0, 0)\) and thickness equal to \(h_0\), and (constant) density \(\rho\) as shown schematically in Fig. 1. Let us take the \( x \)-axis vertically and pointing in the direction of gravity, and the \( y \)-axis directed towards the right, so that the interfaces of the liquid sheet at the nozzle exit are at \(y = \pm h_0/2\). Due to the pressures of the gases on the left and right of the sheet, the liquid may be displaced transversally. The gases surrounding the liquid are assumed to be dynamically passive on account of their small density and dynamic viscosity compared with those of the liquid sheet, and the fluids are assumed to be immiscible.

For two-dimensional, inviscid, liquid sheets, the conservation equations of mass and linear momentum are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho g - \frac{\partial p}{\partial x}, \tag{2}
\]

Fig. 1. Schematic of an inviscid planar liquid sheet.
where $t$ denotes time, $p$ is the pressure, $u$ and $v$ are the velocity components along the $x$- and $y$-axes, respectively, and $g$ is the gravitational acceleration.

These equations are subjected to the following kinematic and dynamic conditions at the sheet’s interfaces

$$v(x, H^+, t) = \frac{\partial H^+}{\partial t} + u(x, H^+, t) \frac{\partial H^+}{\partial x},$$

$$\left( p(x, H^+, t) - p_l \right) J_+^2 = -\frac{\sigma_+ \frac{\partial^2 H^+}{\partial x^2}}{J_+},$$

$$\left( p(x, H^-, t) - p_r \right) J_-^2 = \frac{\sigma_- \frac{\partial^2 H^-}{\partial x^2}}{J_-},$$

where $H^\pm(x, t) = H \pm h/2$ denotes the right (+) and left (−) interfaces, $H$ and $h$ are the transverse displacement and thickness, respectively, of the liquid sheet at $(x, t)$, $p_l$ and $p_r$ denote the pressure of the gases on the left and right, respectively, of the sheet, $\sigma$ is the surface tension, and

$$J_{\pm}^2 = 1 + \left( \frac{\partial H^\pm}{\partial x} \right)^2.$$
these oscillations in an accurate manner, one should solve the two-dimensional compressible flow problems that arise in the two compartments. In this paper, the pressure of the gases on either side of the liquid sheet is assumed to be uniform and this means that the acoustic time is much smaller than both the residence time of the fluid and the characteristic time of forcing.

The length scales that characterize the fluid dynamics of inviscid planar liquid sheets are the thickness at the nozzle exit, $h_0$, the length associated with the gravitational pull, $u_0^2/g$, where $u_0$ denotes a characteristic axial velocity component, and that associated with the surface tension and pressure difference across the sheet, i.e., $\sigma/|\Delta p|$, where $\sigma$ denotes, for example, the arithmetic mean of $\sigma_-$ and $\sigma_+$, and $\Delta p = p_e - p_i$. Moreover, in the absence of forcing or time-dependent conditions at the nozzle exit, the flow is characterized by $\rho$, $u_0$, $\sigma_-$, $\sigma_+$, $g$, $\Delta p$, $h_0$, and $\theta_0$ which denotes the angle with which the liquid sheet exits the nozzle. Therefore, according to $\pi$-Buckingham’s theorem, the solution of Eqs. (1)–(6) should depend on the following dimensionless groups $\theta_0$, $\sigma_-/\sigma_+$, $Fr = u_0^2/gh_0$, $We = pu_0^2h_0/\sigma$, $h_0/L$, and $h_0\Delta p/\sigma$, where $Fr$ and $We$ denote the Froude and Weber numbers, respectively, and $L$ is a characteristic wavelength or length scale in the axial direction. Moreover, the transverse displacement of the liquid sheet depends on the pressure difference across the sheet, and the inertia of the liquid. An estimate of this displacement, i.e., $H_0$, can be obtained from the normal stress conditions, cf. Eq. (5) or Eq. (6), as $H_0/L^2 = O(|\Delta p|/\sigma) + O(h_0/L^2)$. Therefore, it may be concluded that the nondimensional transverse length scales that characterize the sheet are $h_0|\Delta p|/\sigma$ and $h_0/L$, and the sheets are considered as either thin or thick if either $h_0 \ll H_0$ or $h_0 \gg H_0$, respectively; the sheets are considered slender if $h_0 \ll L$.

In the next sections, a variety of models based on perturbation methods, Taylor series expansions of the dependent variables and integral formulations of Eqs. (1)–(6) are developed.

3. Potential flow models based on perturbation methods

For two-dimensional, (constant) density, inviscid, irrotational liquid sheets, Eqs. (1)–(3) reduce to the Laplace equation for the velocity potential and the Bernoulli equation, i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

$$p = \rho \left( gx - \frac{\partial \phi}{\partial t} - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial y} \right)^2 \right).$$
where the function of time that appears in the Bernoulli equation has been included in the velocity potential, i.e., \( \phi, u = \partial \phi / \partial x \) and \( v \partial \phi / \partial y \). Eqs. (8) and (9) are subject to the kinematic and dynamic boundary conditions given by Eqs. (4)–(6).

If \( t, x, y, H, h, \phi \) and \( p \) are nondimensionalized with respect to \( L / u_0, L, h_0, h_0, u_0 L \) and \( \frac{1}{2} \rho u_0^2 \), respectively, one can easily obtain the following nondimensional equations

\[
\epsilon^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{10}
\]

\[
p = \frac{x}{Fr} - 2 \frac{\partial \phi}{\partial t} - \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{\epsilon^2} \left( \frac{\partial \phi}{\partial y} \right)^2, \tag{11}
\]

\[
\frac{\partial \phi}{\partial y}(x, H^\pm, t) = \epsilon^2 \frac{\partial H^\pm}{\partial t} + \epsilon^2 \frac{\partial H^\pm}{\partial x} \frac{\partial \phi}{\partial x}(x, H^\pm, t), \tag{12}
\]

\[
p_h - \left( \frac{x}{Fr} - 2 \frac{\partial \phi}{\partial t} - \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{\epsilon^2} \left( \frac{\partial \phi}{\partial y} \right)^2 \right) = \epsilon^2 \frac{\partial^2 H^+}{\partial x^2}, \quad \text{at } y = H^+, \tag{13}
\]

\[
p_e - \left( \frac{x}{Fr} - 2 \frac{\partial \phi}{\partial t} - \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{\epsilon^2} \left( \frac{\partial \phi}{\partial y} \right)^2 \right) = - \epsilon^2 \frac{\partial^2 H^-}{\partial x^2}, \quad \text{at } y = H^-, \tag{14}
\]

where for the sake of convenience we have used the same symbols to denote dimensional and dimensionless variables,

\[
J^2_\pm = 1 + \epsilon^2 \left( \frac{\partial H^\pm}{\partial x} \right)^2, \tag{15}
\]

\( Fr = u_0^2 / 2 g L, \ We = \rho u_0^2 h_0 / 2 \sigma, \ \epsilon = h_0 / L \ll 1, \) and we have set \( \sigma_- = \sigma_+ = \sigma. \)

For \( H_0 = O(h_0), \) i.e., for transverse displacements on the order of the sheet’s thickness or nearly vertical sheets, and \( \epsilon \ll 1, \) i.e., for slender sheets, substitution of the asymptotic expansion

\[
\psi = \psi_0 + \epsilon^2 \psi_2 + O(\epsilon^4), \tag{16}
\]

where \( \psi \) stands for \( \phi, p \) and \( H^\pm, \) into Eqs. (8) and (9) yields

\[
\phi_0 = Ay + B, \quad \phi_2 = - \frac{1}{6} \nu^3 \frac{\partial^2 A}{\partial x^2} - \frac{1}{2} \nu^3 \frac{\partial^2 B}{\partial x^2} + Cy + D \tag{17}
\]

while the kinematic conditions (cf. Eq. (12)) imply that

\[
\frac{\partial \phi_0}{\partial y}(x, H_0^\pm, t) = 0, \tag{18}
\]
\[
\frac{\partial \phi_2}{\partial y}(x, H_0^+, t) = \frac{\partial H_0^+}{\partial t} + \frac{\partial H_0^+}{\partial x} \frac{\partial \phi_0}{\partial x}(x, H_0^+, t),
\]

(19)

where \(A, B, C \) and \(D \) are functions of \(x\) and \(t\).

Eqs. (17)–(19) yield \(A = 0\) and

\[
C = \frac{\partial H_0^+}{\partial t} + \frac{\partial}{\partial x} \left(H_0^+ \frac{\partial B}{\partial x}\right),
\]

(20)

\[
\frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x} \left(h_0 \frac{\partial B}{\partial x}\right) = 0,
\]

(21)

where \(h_0 = H_0^+ - H_0^-\) is the sheet’s thickness at leading order, and the leading-order axial velocity component is \(\frac{\partial B}{\partial x}\).

The dynamic boundary conditions, i.e., Eqs. (13) and (14), provide two different flow regimes, i.e., the inertia and capillary regimes, which correspond to \(We = O(1)\) and \(We = O(\epsilon^2)\), respectively, as indicated in the next paragraphs.

**Inertia regime**: \(We = O(1)\). If \(We = O(1)\), the dynamic boundary conditions at \(O(\epsilon^0)\) imply that

\[
\frac{\partial \phi_0}{\partial y}(x, H_0^+, t) = 0
\]

(22)

and, at \(O(\epsilon^2)\),

\[
p_t - \frac{\chi}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = 0, \quad \text{at } y = H_0^+, \tag{23}
\]

\[
p_e - \frac{\chi}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = 0, \quad \text{at } y = H_0^- \tag{24}
\]

which imply that \(p_t = p_e\), and this is consistent with the assumption that \(H_0 = O(h_0)\).

There are three equations, i.e., Eqs. (20), (21) and (23), for the four unknowns \(B, C\) and \(H_0^-\), and, in order to close this system of equations, we have to proceed to \(O(\epsilon^4)\) in the asymptotic expansion. At this order, the dynamic boundary conditions yield

\[
2 \frac{\partial \phi_2}{\partial t} + 2 \frac{\partial \phi_0}{\partial t} + \frac{\partial \phi_2}{\partial x} \frac{\partial B}{\partial x} + \left(\frac{\partial \phi_2}{\partial y}\right)^2 = \frac{1}{We} \frac{\partial^2 H_0^+}{\partial x^2}, \quad \text{at } y = H_0^+, \tag{25}
\]

\[
2 \frac{\partial \phi_2}{\partial t} + 2 \frac{\partial \phi_0}{\partial t} + \frac{\partial \phi_2}{\partial x} \frac{\partial B}{\partial x} + \left(\frac{\partial \phi_2}{\partial y}\right)^2 = -\frac{1}{We} \frac{\partial^2 H_0^-}{\partial x^2}, \quad \text{at } y = H_0^- \tag{26}
\]
which, upon using Eq. (17), become
\[
\frac{\partial}{\partial t} \left(-B'H_0^2 + 2CH_0 + 2D\right) + B'(-B''H_0^2 + 2C'H_0 + 2D') + (C - B''H_0^2)^2 = -\frac{1}{We} \frac{\partial^2 H_0^-}{\partial x^2}, \tag{27}
\]
\[
\frac{\partial}{\partial t} \left(-B'H_0^2 + 2CH_0 + 2D\right) + B'(-B''H_0^2 + 2C'H_0 + 2D') + (C - B''H_0^2)^2 = \frac{1}{We} \frac{\partial^2 H_0^+}{\partial x^2}, \tag{28}
\]
where the primes denote differentiation with respect to \(x\).

Eqs. (27) and (28) together with Eqs. (20), (21) and (23) provide a closed system of equations for \(H_0, h_0, C, B\) and \(D\). Note that the leading-order axial velocity is \(u_0 = \partial \phi_0 / \partial x = \partial B / \partial x\), Eq. (21) is the continuity equation at leading order, and differentiation of Eq. (23) with respect to \(x\) yields
\[
\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = \frac{1}{2Fr}. \tag{29}
\]

**Capillary regime:** \(We = O(e^2)\). If \(We = We^2\), where \(W = O(1)\), use of Eq. (16) yields Eqs. (17)–(19), and the dynamic boundary conditions at \(O(e^2)\) are
\[
p_e - \frac{x}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = -\frac{1}{W} \frac{\partial H_0^-}{\partial x^2}, \tag{30}
\]
\[
p_h - \frac{x}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = \frac{1}{W} \frac{\partial^2 H_0^+}{\partial x^2}, \tag{31}
\]
which upon addition and subtraction yield
\[
p_h + p_e - \frac{2x}{Fr} + 4 \frac{\partial B}{\partial t} + 2 \left(\frac{\partial B}{\partial x}\right)^2 = \frac{1}{W} \frac{\partial^2 h_0}{\partial x^2}, \tag{32}
\]
\[
p_h - p_e = \frac{2}{W} \frac{\partial^2 H_0}{\partial x^2}. \tag{33}
\]
Eq. (33) can be integrated analytically if the gases on both sides of the liquid sheet are dynamically passive, i.e., if \(p_t = p_t(t)\) and \(p_e = p_e(t)\), and its solution is
\[
H_0 = -\frac{W \Delta p}{4} x^2 + \alpha(t) x + \beta(t), \tag{34}
\]
where \(\alpha = 0\) because \(H_0(0, t) = 0\) and \(\beta\) can be determined from Eq. (20) applied at the nozzle exit. It should be pointed out that, in general, \(p_t\) and \(p_e\) are functions of space and time due to the acoustic waves generated in the gases when the liquid sheet undergoes any transverse displacement or local thickness variation, and the fluid dynamics of the liquid are coupled to those of the gases that surround it. Moreover, the space dependence of the gas pressure can only
be neglected as a first-order approximation when the acoustic time in the gas is smaller than both the residence time and the characteristic time of forcing.

Eq. (20) can also be written as

$$C = \frac{\partial H_0}{\partial t} + \frac{\partial B}{\partial x} \frac{\partial H_0}{\partial x}$$  \hspace{1cm} (35)$$

while differentiation of Eq. (32) with respect to \(x\) yields

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = \frac{1}{2Fr} + \frac{1}{4W} \frac{\partial^3 h_0}{\partial x^3}$$  \hspace{1cm} (36)$$

which together with Eq. (21) correspond to the thickness problem.

Eqs. (21) and (36) provide the dynamics of the sheet’s thickness, \(h_0\), and the liquid’s axial velocity component at leading order; these equations are decoupled from the sheet’s transverse displacement, \(H_0\), which is governed by Eq. (34).

4. Potential flow models of thin liquid sheets

In Section 3, it was assumed that the characteristic transverse displacement of the liquid sheet was of the same order as the sheet’s thickness, i.e., the liquid sheet was almost vertical. In this section, it is assumed that the sheet is both slender and thin, i.e., \(\varepsilon \ll 1\) and \(h_0/H_0 = \mu \ll 1\), and we consider the case where \(\mu = \varepsilon^2\). Under these conditions, Eq. (16) is still valid for \(\phi, \rho \) and \(H^\pm\) but here \(H_0^+ = H_0^-\) and \(h = \varepsilon^2 h_2 + O(\varepsilon^4)\). Use of Eq. (16) in Eqs. (10)–(14) yields Eq. (17) and

$$\frac{\partial \phi_0}{\partial y}(x, H_0, t) = 0, \hspace{1cm} (37)$$

$$\frac{\partial \phi_2}{\partial y}(x, H_0, t) = \frac{\partial H_0}{\partial t} + \frac{\partial H_0}{\partial x} \frac{\partial \phi_0}{\partial x}(x, H_0, t) \hspace{1cm} (38)$$

which imply that \(A = 0\),

$$C = \frac{\partial H_0}{\partial t} + \frac{\partial}{\partial x} \left( H_0 \frac{\partial B}{\partial x} \right), \hspace{1cm} (39)$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x} \left( h_2 \frac{\partial B}{\partial x} \right) = 0 \hspace{1cm} (40)$$

at \(O(\varepsilon^0), O(\varepsilon^2)\) and \(O(\varepsilon^4)\), respectively.

The assumption \(\mu = \varepsilon^2\) is not necessary as one can easily show by taking into account that, for thin sheets, \(|\partial/\partial y| \gg |\partial/\partial x|\) and performing an asymptotic analysis similar to the ones carried out in previous paragraphs. However, this assumption is employed here to illustrate the differences between
one-dimensional models based on perturbation methods and those based on other techniques.

As in Section 3, two flow regimes may be identified depending on the magnitude of the Weber number.

**Inertia regime:** \( \text{We} = \mathcal{O}(1) \). The dynamic boundary conditions for \( \text{We} = \mathcal{O}(1) \) yield \( p_i = p_e \) at leading order and Eq. (29) at \( \mathcal{O}(\epsilon^2) \). At \( \mathcal{O}(\epsilon^4) \), the dynamic boundary conditions result in

\[
\frac{\partial}{\partial t} \left(-B'H_0^2 + 2CH_0 + 2D\right) + B' \left(-B''H_0^2 + 2C'H_0 + 2D'\right) + (C - B'H_0)^2 = -\frac{1}{\text{We}} \frac{\partial^2 H_0}{\partial x^2},
\]

and

\[
\frac{\partial}{\partial t} \left(-B'H_0^2 + 2CH_0 + 2D\right) + B' \left(-B''H_0^2 + 2C'H_0 + 2D'\right) + (C - B'H_0)^2 = \frac{1}{\text{We}} \frac{\partial^2 H_0}{\partial x^2}
\]

which imply that \( \partial^2 H_0 / \partial x^2 = 0 \). Therefore, Eqs. (29) and (40) govern the fluid dynamics of slender, thin, planar liquid sheets at \( \text{We} = \mathcal{O}(1) \). \( C, D \) and \( H_0 \) can be calculated from Eqs. (39), (41) and (42).

**Capillary regime:** \( \text{We} = \mathcal{O}(\epsilon^2) \). The dynamic conditions for \( \text{We} = \mathcal{O}(\epsilon^2) \) yield at \( \mathcal{O}(\epsilon^2) \)

\[
p_i - \frac{x}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = -\frac{1}{W} \frac{\partial^2 H_0}{\partial x^2},
\]

and

\[
p_i - \frac{x}{Fr} + 2 \frac{\partial B}{\partial t} + \left(\frac{\partial B}{\partial x}\right)^2 = \frac{1}{W} \frac{\partial^2 H_0}{\partial x^2}
\]

which upon addition and subtraction yield

\[
p_i + p_e = \frac{2x}{Fr} + 4 \frac{\partial B}{\partial t} + 2 \left(\frac{\partial B}{\partial x}\right)^2 = 0,
\]

and

\[
p_i - p_e = \frac{2}{W} \frac{\partial^2 H_0}{\partial x^2}.
\]

Furthermore, differentiation of Eq. (44) with respect to \( x \) yields Eq. (29). The solution of Eq. (46) is Eq. (34), and Eqs. (29) and (40) are to be solved to determine \( B \) and \( h_2 \); therefore, the thickness and transverse displacement problems are decoupled for slender, thin liquid sheets at \( \text{We} = \mathcal{O}(\epsilon^2) \).

It must be stated that the asymptotic equations for slender liquid sheets developed in Sections 3 and 4 are not valid near the nozzle exit where \( H(0, t) = 0 \) due to the relaxation of the velocity profile from the no-penetration conditions at the nozzle’s walls to the free-surface conditions below the nozzle. It should also be noted that the one-dimensional flow models developed in
Sections 3 and 4 have analytical solutions under steady state conditions whose linear and nonlinear temporal stability can be analyzed in closed form in terms of the thickness \[20\]. Such stability analyses indicate that the varicose or symmetric mode is stable.

5. Approximate models

In the two previous sections, one-dimensional equations for the fluid dynamics of inviscid, irrotational, planar liquid sheets were derived by means of perturbation methods based on the slenderness ratio. In this section, we present one-dimensional models based on integral formulations, variational principles and Taylor’s series expansions.

5.1. Models based on integral formulations

Integration of Eqs. (1)–(3) from \(H^-\) to \(H^+\) and use of the kinematic and normal stress or dynamic boundary conditions at the sheet’s interfaces and Leibnitz rule yield

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_{H^-}^{H^+} u \, dy = 0, \tag{47}
\]

\[
\frac{\partial}{\partial t} \int_{H^-}^{H^+} u \, dy + \frac{\partial}{\partial x} \int_{H^-}^{H^+} u^2 \, dy = -\frac{1}{\rho} \frac{\partial}{\partial x} \int_{H^-}^{H^+} \rho \, dy + gh + \frac{1}{\rho} \left( p_e - \frac{\sigma_+}{J^+_+} \frac{\partial^2 H^+}{\partial x^2} \right) \frac{\partial H^+}{\partial x} - \frac{1}{\rho} \left( p_i + \frac{\sigma_-}{J^-_-} \frac{\partial^2 H^-}{\partial x^2} \right) \frac{\partial H^-}{\partial x}, \tag{48}
\]

\[
\frac{\partial}{\partial t} \int_{H^-}^{H^+} v \, dy + \frac{\partial}{\partial x} \int_{H^-}^{H^+} uv \, dy = \frac{1}{\rho} \left( p_e - p_i + \frac{\sigma_+}{J^+_+} \frac{\partial^2 H^+}{\partial x^2} + \frac{\sigma_-}{J^-_-} \frac{\partial^2 H^-}{\partial x^2} \right), \tag{49}
\]

where \(h = H^+ - H^-\) is the local liquid sheet’s thickness.

Taylor series expansions of the (two) kinematic conditions about \(v = H\) yield

\[
v(x, H, t) = O(h) = \frac{\partial H}{\partial t} + u(x, H, t) \frac{\partial H}{\partial x} + O \left( h, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x} \right) \tag{50}
\]

which implies, upon subtraction, that, to \(O(h^2, (\partial h/\partial x)^2)\),

\[
v(x, H, t) = \frac{\partial H}{\partial t} + u(x, H, t) \frac{\partial H}{\partial x}. \tag{51}
\]
In order to develop one-dimensional models based on Eqs. (47)–(50), one can use the Taylor expansions of $u$ and $p$ about, for example, $y = H$, integrate the continuity equation to obtain $v$, and the results can then be substituted into Eqs. (47)–(49). This substitution would provide series in powers of $h$, and by equating terms of the same power of $h$, one could obtain a hierarchy of equations for the Taylor’s series coefficients which are only functions of $x$ and $t$; note that the use of Taylor series expansions for $u$, $v$ and $p$ about $y = H$ allows to perform the integrals in Eqs. (47)–(49) analytically. Moreover, the Taylor series expansion can also be substituted into Eq. (4) to obtain a hierarchy of equations which would provide the kinematic conditions at different orders of $h$. Since methods based on Taylor series expansions will be treated in Section 5.3, they shall not be considered any further in this section. However, it is worth indicating that the convergence of this technique depends on $h$.

If $u(x,y,t)$ is assumed to be uniform across the liquid sheet, i.e., $u(x,y,t) = u_0(x,t)$, then integration of the continuity equation yields

$$v(x,y,t) = v_0(x,t) - (y - H) \frac{\partial u_0}{\partial x}$$  \hspace{1cm} (52)

while, if the pressure is approximated by $p(x,y,t) = p_0(x,t) + (y - H)p_1(x,t)$, then Eqs. (47)–(49) and (51) become

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (u_0 h) = 0,$$ \hspace{1cm} (53)

$$h \left( \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} \right) = -\frac{1}{\rho} \frac{\partial}{\partial x} (p_0 h) + gh + \frac{1}{\rho} \left( p_1 - \sigma_+ \frac{\partial^2 H^+}{\partial x^2} \right) \frac{\partial H^+}{\partial x}$$

$$- \frac{1}{\rho} \left( p_e + \frac{\sigma_-}{J_+} \frac{\partial^2 H^-}{\partial x^2} \right) \frac{\partial H^-}{\partial x},$$  \hspace{1cm} (54)

$$h \left( \frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} \right) = \frac{1}{\rho} \left( p_0 - p_1 + \sigma_+ \frac{\partial^2 H^+}{\partial x^2} + \frac{\sigma_-}{J^-} \frac{\partial^2 H^-}{\partial x^2} \right),$$ \hspace{1cm} (55)

$$v_0 = \frac{\partial H}{\partial t} + u_0 \frac{\partial H}{\partial x}.$$  \hspace{1cm} (56)

The use of the normal stress conditions implies that

$$p_0 = \frac{1}{2} \left( p_1 + p_e - \frac{\sigma_+}{J_+} \frac{\partial^2 H^+}{\partial x^2} + \frac{\sigma_-}{J_-} \frac{\partial^2 H^-}{\partial x^2} \right),$$  \hspace{1cm} (57)

$$p_1 = \frac{1}{h} \left( p_1 - p_e - \frac{\sigma_+}{J_+} \frac{\partial^2 H^+}{\partial x^2} - \frac{\sigma_-}{J_-} \frac{\partial^2 H^-}{\partial x^2} \right).$$  \hspace{1cm} (58)
and \( p_0 \) can be substituted into Eq. (54) to yield

\[
h \left( \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} \right) = gh + \frac{1}{\rho} \left( p_i - p_e - \sigma_+ \frac{\partial^2 H^+}{\partial x^2} - \sigma_- \frac{\partial^2 H^-}{\partial x^2} \right) \frac{\partial H}{\partial x} - \frac{h}{2} \frac{\partial}{\partial x} \left( \sigma_+ \frac{\partial^2 H^+}{\partial x^2} + \sigma_- \frac{\partial^2 H^-}{\partial x^2} \right).
\]

(Eq. (59))

Eqs. (53), (55), (56) and (59) are a system for \( u_0, v_0, H \) and \( h \) which is \( O(h^2) \) accurate.

If \( p \) were approximated by \( p = p_0(x,t) \), then the normal stress conditions would imply

\[
p_i - p_e = \frac{\sigma_+}{J_+} \frac{\partial^2 H^+}{\partial x^2} + \frac{\sigma_-}{J_-} \frac{\partial^2 H^-}{\partial x^2},
\]

(Eq. (60))
i.e., the difference of pressure across the liquid sheet is balanced by surface tension, \( p_0 \) is still given by Eq. (57), and Eq. (59) still applies. Eq. (60) coincides with Eqs. (33) and (46).

The linear pressure approximation employed in deriving Eq. (55) is not strictly correct since, when the linear approximations for the velocity components are substituted into Eqs. (2) and (3), second-degree polynomials in \( y \) result in the left-hand sides of these equations, and, therefore, according to Eqs. (2) and (3), \( p \) should be governed by second- and third-degree polynomials in \( y \), respectively. This inconsistency also characterizes the variational methods presented in Section 5.2. Furthermore, the fact that the continuity and linear momentum equations were integrated in the formulation presented in this section indicates that these equations are satisfied in a weak (or integral) sense, whereas only the linear momentum equations are satisfied in a weak sense in the variational techniques discussed in Section 5.2.

It must be noted that, for thin sheets, i.e., \( |\partial/\partial y| \gg |\partial/\partial x| \) or \( h_0 \ll L \), the continuity equation and the irrotational flow condition imply that, to leading order, \( u \) and \( v \) are only functions of \( x \), whereas expansion of the dynamic boundary conditions around the sheet’s midline yields Eqs. (55) and (59) to leading order.

The derivation presented here was based on dimensional quantities and does not make use of the slenderness parameters. An analogous derivation can easily be performed for the corresponding nondimensional equations.

If \( h_0 \ll H_0 \), i.e., for thin liquid sheets, and \( \sigma_- = \sigma_+ = \sigma \), Taylor series expansions of the right-hand sides of Eqs. (55) and (59) about \( H \) yield at \( O(h_0/H_0)^2 \)

\[
v_0 = \frac{\partial H}{\partial t} + u_0 \frac{\partial H}{\partial x},
\]

(Eq. (61))

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (u_0 h) = 0,
\]

(Eq. (62))
\[ h\left( \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} \right) = gh - \frac{1}{\rho} \left( \Delta p + \frac{2\sigma}{J^3} \frac{\partial^2 H}{\partial x^2} \right) \frac{\partial H}{\partial x}, \] (63)

\[ h\left( \frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} \right) = \frac{1}{\rho} \left( \Delta p + \frac{2\sigma}{J^3} \frac{\partial^2 H}{\partial x^2} \right), \] (64)

where

\[ J^2 = 1 + \left( \frac{\partial H}{\partial x} \right)^2. \] (65)

These equations coincide with those developed in the theories of inviscid annular liquid jets and planar liquid sheets by the author [25] where \( p \) was approximated by \( p_0 \), the terms proportional to \( h \) and \( \partial h/\partial x \) which appear in the right-hand sides of Eqs. (55) and (59) were neglected, i.e., \( \partial h/\partial x \) was neglected in comparison with \( \partial H/\partial x \). The equations are not valid near the nozzle exit where \( H(0,t) = 0 \) and the assumption \( h_0 \ll H_0 \) is violated.

The model represented by Eqs. (61)–(65) coincides with the hydraulic model of Keller and Weitz [10] who considered \( p_e = p_i \), and is further analyzed in Section 6. The hydraulic theory of jets is based on the assumption that each particle of the jet moves independently along a parabolic trajectory.

### 5.2. Variational methods

Another technique which may be used to develop approximate equations for the fluid dynamics of planar liquid sheets is based on the method of weighted residuals [8,21,26]. First, one introduces the mapping \((x,y,t) \rightarrow (z,s,t)\) where \( z = x \), \( \tau = t \) and \( s = 2(y - H)/h \), so that the sheet’s interfaces are located at \( s = \pm 1 \); then the velocity components are expanded as

\[ u = \sum_{n=0}^{M} U_n(x,t)s^n, \quad v = \sum_{n=0}^{N} V_n(x,t)s^n, \] (66)

where the natural numbers \( M \) and \( N \) could be different, although here it will be assumed that \( M = N = K \), and the coefficients \( U_n \) and \( V_n \) are related through the continuity equation and kinematic conditions; note that the continuity equation is a kinematic relationship. Substitution of Eq. (66) into the longitudinal and transverse momentum equations, i.e., Eqs. (2) and (3) implies that the polynomials in \( s \) that appear in the left-hand sides of these equations are of degrees equal to \( 2K \). Therefore, if an analogous expansion to Eq. (66) were used for the pressure, the degree of this polynomial would be equal to \( 2K \) according to the longitudinal momentum equation and \( 2K + 1 \) according to the transverse momentum equation. This contradiction is resolved by employing the weak variational formulation due to Kantorovich and Krylov [26] whereby
the momentum equations are multiplied by $s^n$ and integrated across the liquid sheet, i.e., from $s = -1$ to 1 for $n = 0, 1, \ldots, K$, and this weak variational principle provides $(4K + 2)$ equations which together with the continuity equation and the kinematic condition provides a hierarchy of equations for $U_n$, $V_n$ and the moments of the pressure, i.e.,

$$P_n = \int_{-1}^{1} ps^n ds. \quad (67)$$

This method is conceptually simple, but very tedious, because the metric of the transformation is a function of $(x, y, t)$. Moreover, only the linear momentum equations are treated in a weighted residual manner, whereas the kinematic conditions and the continuity equation are dealt with in a classical sense. Rather than carrying out the procedure just described in full detail for arbitrary $K$, we will analyze the approximation $u = U_0(z, t)$ and $v = V_0(z, t) + sV_1(z, t)$, for which the kinematic conditions imply

$$2V_1 = \frac{\partial h}{\partial t} + U_0 \frac{\partial h}{\partial z}, \quad (68)$$

$$V_0 = \frac{\partial H^-}{\partial t} + U_0 \frac{\partial H^-}{\partial z} \quad (69)$$

while the continuity equation yields

$$V_1 = -\frac{h}{2} \frac{\partial U_0}{\partial z} \quad (70)$$

which combined with Eq. (68) yields

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}(U_0 h) = 0. \quad (71)$$

Multiplication of the longitudinal momentum equation by $s^0$ and integration of the resulting equation yields

$$h \left( \frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial z} \right) = gh + \frac{1}{\rho} \left( p(x, H^+, t) \frac{\partial H^+}{\partial x} - p(x, H^-, t) \frac{\partial H^-}{\partial x} - \frac{1}{2} \frac{\partial}{\partial z} (p_0 h) \right) \quad (72)$$

while multiplication of the transverse momentum equation by $s^0$ and $s^1$ and integration of the resulting equations yields

$$h \left( \frac{\partial V_0}{\partial t} + U_0 \frac{\partial V_0}{\partial z} \right) = p(x, H^-, t) - p(x, H^+, t), \quad (73)$$

$$h \left( \frac{\partial V_1}{\partial t} + U_0 \frac{\partial V_1}{\partial z} \right) = 3(p(x, H^-, t) - p(x, H^+, t) + p_0), \quad (74)$$
and Eqs. (69)–(74) form a closed system for $U_0$, $V_0$, $V_1$, $P_0$, $H$ and $h$ which are functions of $x$ and $t$.

Note that, if in Eq. (67), $p$ were approximated by $p_0(z, \tau) + sp_1(z, \tau)$, then $P_0 = 2p_0$ and Eq. (72) would then be identical to Eq. (59). This result is not surprising, for the integral formulation presented in Section 5.1 is really a variational formulation for the longitudinal and transverse momentum equations for $K = 0$. Note also that $P_0$ can be eliminated in terms of the sheet’s geometry and pressures at the sheet’s interfaces to obtain an equation that depends on $U_0$, $V_0$ and $V_1$, and that $V_1$ can be eliminated by using Eq. (70).

The integral formulation methods and variational techniques presented in Sections 5.1 and 5.2 satisfy the governing equations in a weak sense, whereas the perturbation expansions of Sections 3 and 4 result in a hierarchy of equations at different orders in the slenderness ratio, and such a hierarchy provides the dependence of the pressure and velocity fields on $x$, $y$ and $t$. By way of comparison, the dependence of the velocity and pressure on the transverse coordinate in integral formulation methods and variational techniques is assumed a priori.

5.3. Methods based on series expansions

These methods are analogous to the variational ones described in Section 5.2 and are based on the substitution of the following expansions

$$u = \sum_{n=0}^{K} u_n(x, t)(y - H)^n, \quad v = \sum_{n=0}^{K} v_n(x, t)(y - H)^n,$$

$$p = \sum_{n=0}^{K} p_n(x, t)(y - H)^n \tag{75}$$

into the differential equations for conservation of mass and linear momentum, and the kinematic and dynamic boundary conditions. This substitution results in series that depend on $(y - H)$ for the continuity and momentum equations, and on $h$ for the kinematic and dynamic boundary conditions [8,21]. By setting the coefficients of the monomials in $(y - H)$ or $h$ to zero, a hierarchy of equations is obtained, and these equations reduce to those derived in Section 5.1 for $K = 1$ and, therefore, are not repeated here. Therefore, the accuracy of these methods for $K = 1$ is $O(h^2)$ which is the next term neglected in the series expansion. By way of contrast, if Eq. (75) with $K = 1$ were employed in Eqs. (47)–(49), one would obtain an accuracy of $O(h^3)$. However, the kinematic condition will still be valid to $O(h^2)$ on account of the Taylor’s series expansion used.

As indicated above, one-dimensional models based on Taylor’s series expansion may coincide with those derived from variational or integral formu-
lations. However, the latter contain second-order spatial derivatives for the transverse displacement of the sheet’s centerline, whereas the leading-order equations for slender sheets obtained from perturbation methods do not account for pressure differences and surface tension effects (cf. Eq. (29)), and those for slender and thin sheets contain third-order derivatives with respect to the sheet’s thickness (cf. Eq. (36)); the appearance of this third-order derivative is due to the scaling of the surface tension terms. Moreover, the leading-order equations provided by perturbation methods govern the axial velocity component and the thickness, whereas the transverse displacement and transverse velocity components are uncoupled and can be obtained once the axial velocity component and the thickness are calculated. By way of contrast, the equations obtained from the integral formulation, i.e., Eqs. (61)–(65), couple the axial and transverse velocity components, the thickness and the transverse displacement, except when \( H = 0 \), i.e., for vertically falling sheets, or as, indicated in Section 6, whenever the Weber number is very large. Although the one-dimensional equations obtained from perturbation methods are asymptotically valid, it is convenient to determine the conditions under which the approximate models derived in this paper are asymptotically equivalent to the exact equations for liquid membranes. This is the subject of Section 6.

6. Approximate nondimensional equations

Since the approximate methods presented in Sections 5.1–5.3 result in similar sets of equations, we shall only considered the model given by Eqs. (61)–(65) presented in Section 5.1. If time, velocity components and lengths are nondimensionalized with respect to \( h_0/u_0 \), \( u_0 \) and \( h_0 \), respectively, then Eqs. (61)–(65) can be written as

\[
v_0 = \frac{\partial H}{\partial t} + u_0 \frac{\partial H}{\partial x}, \tag{76}
\]

\[
h \left( \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} \right) = \frac{h}{Fr} \left( - C_{pn} + \frac{1}{J^3} \frac{\partial^2 H}{\partial x^2} \right) \frac{\partial H}{\partial x}, \tag{78}
\]

\[
h \left( \frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} \right) = \frac{1}{We} \left( - C_{pn} + \frac{1}{J^3} \frac{\partial^2 H}{\partial x^2} \right) \tag{79}
\]

and Eq. (65), where \( C_{pn} = -\Delta p h_0/2\sigma \), \( Fr = u_0^2/gh_0 \) and \( We = \rho u_0^2 h_0/2\sigma \).
6.1. Comparison with liquid membranes

Eqs. (76)–(79) are very similar to those for planar liquid membranes [21], i.e., liquid sheets of zero thickness, which are governed by the following equations

\[ \frac{\partial m}{\partial t} + u \frac{\partial m}{\partial x} + m \cos \theta \left( \frac{\partial u}{\partial x} \cos \theta + \frac{\partial v}{\partial x} \sin \theta \right) = 0, \quad (\text{80}) \]

\[ v = \frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x}, \quad (\text{81}) \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \sin \theta \left( \frac{\partial p}{m} + 2\sigma \cos^3 \theta \frac{\partial^2 H}{\partial x^2} \right) + g, \quad (\text{82}) \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = \cos \theta \left( \frac{\partial p}{m} + 2\sigma \cos^3 \theta \frac{\partial^2 H}{\partial x^2} \right), \quad (\text{83}) \]

where \( m \) is the mass per unit area and \( \tan \theta = \partial H/\partial x \) is the membrane’s slope.

If \( m, t, u, v, H \) and \( x \) are nondimensionalized with respect to \( m_0, L/u_0, u_0, \varepsilon u_0, H_0 \) and \( L \), respectively, where \( \varepsilon^2 = H_0/L \), Eqs. (80)–(83) become

\[ \frac{\partial m}{\partial t} + u \frac{\partial m}{\partial x} + m \cos \theta \left( \frac{\partial u}{\partial x} \cos \theta + \varepsilon^2 \frac{\partial v}{\partial x} \sin \theta \right) = 0, \quad (\text{84}) \]

\[ v = \frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x}, \quad (\text{85}) \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{\sin \theta}{\varepsilon^2 m W} \left( - C_{pn} + \cos^3 \theta \frac{\partial^2 H}{\partial x^2} \right) + \frac{1}{\varepsilon^2 F_r}, \quad (\text{86}) \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = \frac{\cos \theta}{\varepsilon^2 m W} \left( - C_{pn} + \cos^3 \theta \frac{\partial^2 H}{\partial x^2} \right), \quad (\text{87}) \]

where

\[ \sin \theta = \frac{\varepsilon^2 (\partial H/\partial x)}{\left( 1 + \varepsilon^4 (\partial H/\partial x)^2 \right)^{1/2}}, \quad \cos \theta = \frac{1}{\left( 1 + \varepsilon^4 (\partial H/\partial x)^2 \right)^{1/2}}, \quad (\text{88}) \]

\( C_{pn} = -\Delta p H_0/2\sigma, \quad F_r = u_0^2/g H_0 \) and \( W = \rho u_0^2 H_0/2\sigma \).

If \( W = \varepsilon^{-4} W, \quad C_{pn} = O(1) \) and \( F_r = \varepsilon^{-2} F, \) where \( W = O(1) \) and \( F = O(1) \), then an asymptotic expansion identical to Eq. (16) for \( m, u, v \) and \( H \) indicates that, to leading order, the governing equations are

\[ \frac{\partial m_0}{\partial t} + \frac{\partial}{\partial x} (m_0 u_0) = 0, \quad (\text{89}) \]
Eqs. (76)–(79) for planar liquid sheets are identical to Eqs. (89)–(92) provided that $h$ is interpreted as mass per unit and the second term in the right-hand side of Eq. (78) is neglected, i.e., the slope must be small. However, one should not conclude that similar equations govern the fluid dynamics of both sheets and membranes, for Eqs. (80)–(83) are exact, whereas the derivation of Eqs. (76)–(79) was based on some approximations to the pressure and velocity fields and $h_0 \ll H_0$; in addition, planar liquid membranes only have a free surface, whereas planar liquid sheets have two.

It should be noted that the one-dimensional equations for inviscid, planar liquid membranes have analytical solutions under steady state conditions whose linear temporal stability can be analyzed in closed form in terms of the transverse displacement [21]. Such a stability analysis indicates that the varicose or symmetric mode is stable.

It should also be pointed out that asymptotic methods based on the slenderness and thickness of the liquid sheet may be used in Eqs. (76)–(79) to determine the leading-order equations of slender almost vertically falling sheets, and thin and slender planar liquid sheets. Such an asymptotic analysis will not be presented here.

6.2. Approximate analytical solutions of steady integral formulation models

For steady, slender, thin liquid sheets, i.e., $h_0 \ll H_0 \ll L$, Eqs. (77) and (78) may be approximated by

$$\begin{align*}
  v_0 &= \frac{\partial H_0}{\partial t} + u_0 \frac{\partial H_0}{\partial x}, \\
  \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} &= \frac{1}{Fr}, \\
  \frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} &= \frac{1}{m_0 W} \left( -C_p + \frac{\partial^2 H_0}{\partial x^2} \right),
\end{align*}$$

at large Weber numbers, and, thus,

$$u_0^2 = 1 + \frac{2x}{Fr},$$

where we have assumed that $u_0(0) = 1$.

For large Weber numbers and $C_p = O(1)$, the right-hand side of Eq. (79) may be set to zero, $v_0 = \gamma$ where $\gamma$ is a constant, and Eq. (76) can be integrated to yield

$$H(x) = \gamma u_0 Fr + \delta$$
which states that $H$ increases as $x^{1/2}$ for large $x$, and $\delta$ is a constant. If this equation were applied at the nozzle exit where $H(0) = 0$ then $\delta = -\gamma Fr$, and the condition that $dH/dx \ll 1$ implies that $\gamma/u_0 \ll 1$. Moreover, since $h = 1/u_0$, $h/H \ll 1$ provided that $\gamma \gg 1$; therefore, $dH/dx \ll 1$ and $h/H \ll 1$ cannot be satisfied simultaneously at the nozzle exit.

The approximate analytical solutions just obtained do not involve surface tension and the pressure difference across the liquid sheet. The pressure difference may be accounted for by approximating Eq. (79) by

$$\frac{dv_0}{dx} = -\frac{1}{We} C_{pn}$$

which is valid for slender liquid sheets with $We \gg 1$ and $C_{pn} = O(We)$, and can be integrated to yield

$$v_0 = -\frac{1}{We} C_{pn} x + x,$$

where $x$ is an integration constant. Furthermore, by using the kinematic condition, i.e., Eq. (76), and Eq. (97), one can readily obtain

$$H = Fr \left(-\frac{C_{pn} Fr}{2 We} \left(\frac{u_0^3}{3} - u_0\right) + x u_0\right) + \beta,$$

where $\beta$ is another integration constant.

Both pressure differences and surface tension effects may be accounted for if Eq. (99) is approximated, for $We = O(1)$ and $C_{pn} = O(1)$, by

$$\frac{dv_0}{dx} = -\frac{1}{We} \left(C_{pn} + \frac{d^2H}{dx^2}\right)$$

which is valid for slender liquid sheets. Using Eqs. (76) and (93) in Eq. (99), one can readily integrate the resulting equation and obtain

$$H = v + x Fr \left(u_0 + \frac{1}{We} \ln \left|u_0 - \frac{1}{We}\right|\right) - \frac{C_{pn} Fr^2}{2 We} \left(\frac{u_0^3}{3} - u_0 + \frac{1}{We} \left(\frac{u_0^2}{2} - u_0\right)\right) + 2 \frac{u_0}{We^2} + \frac{1}{We} \left(\frac{1}{We^2} - 1\right) \ln \left|u_0 - \frac{1}{We}\right|,$$

which is valid for $We \neq 1$ and where $x$ and $v$ are integration constants. The case $We = 1$ will be considered in Section 6.3.

For the sake of completeness, we also show here the pressurized, i.e., $C_{pn} \neq 0$, hydrostatic capillary solution corresponding to zero gravity and velocity, which may be written as

$$(H + \beta)^2 + (x + x)^2 = \frac{1}{C_{pn}^2}$$
which is a circumference, and $\alpha$ and $\beta$ are constants of integration. This solution corresponds to velocities at the nozzle exit which are much smaller than the capillary velocity, i.e., $(2\sigma/\rho h_0)^{1/2}$, and may be easily obtained by nondimensionalizing Eqs. (76)–(79) and setting the velocity equal to zero.

For nonzero gravity, the hydrostatic capillary solution is

$$H = Fr \arcsin z - 2Fr^2 C_{pn} Q,$$

where

$$z = Fr \left( -C_{pn} + \alpha \exp \left( \frac{x}{Fr} \right) \right)$$

and

$$Q = -\frac{1}{1 + Fr C_{pn} Q},$$

if $(-Fr C_{pn})^2 = 1$,

$$Q = \frac{1}{\left( \left( -Fr C_{pn} \right)^2 - 1 \right)^{1/2}} \arctan \frac{1 + Fr C_{pn} Q}{\left( \left( -Fr C_{pn} \right)^2 - 1 \right)^{1/2}},$$

if $(-Fr C_{pn})^2 > 1$,

$$Q = \frac{1}{\left( 1 - \left( -Fr C_{pn} \right)^2 \right)^{1/2}} \arctanh \frac{1 + Fr C_{pn} Q}{\left( 1 + \left( Fr C_{pn} \right)^2 \right)^{1/2}},$$

if $(-Fr C_{pn})^2 < 1$, where $q = \tan(\psi/2)$ and $\sin \psi = z$.

The hydrostatic solutions just presented can be readily obtained by nondimensionalizing the time, velocity components, lengths and pressure in Eqs. (76)–(79) with respect to $H_0/u_0$, $u_0$, $H_0$ and $2\sigma/H_0$, respectively, where $u_0 = (2\sigma/H_0)^{1/2}$ is the capillary velocity. This together with the condition that the capillary velocity is larger than the liquid velocity at the nozzle exit permits to use an asymptotic expansion whose leading-order equations coincide with Eq. (102).

### 6.3. Steady state singularities

Eqs. (76)–(79) and (65) for liquid sheets and Eqs. (80)–(83) for liquid membranes may show singularities under steady state conditions. The singularities of planar liquid membranes have been previously analyzed by the author by means of differential and algebraic approaches [17]. In this paper, these two approaches are used to analyze the singularities of steady planar liquid sheets.
Eqs. (76)–(79) can be re-arranged and written as
\[
\left( \frac{\cos \theta}{We} - u \right) \frac{d^2 H}{dx^2} = \frac{1}{u Fr} \tan \theta + C_{pn} \frac{1}{We \cos^2 \theta}
\]  
(107)
and the left-hand side of this equation is nil whenever \( \cos \theta = u We \). If this condition is satisfied and the right-hand side of the equation is not nil, then the curvature of the liquid sheet is undefined and this is not physically possible. Therefore, whenever \( \cos \theta = u We \), the right-hand side of Eq. (107) must be also nil, i.e., the singularity is removable, and this condition implies that
\[
\sin \theta = -C_{pn} \frac{Fr}{We^2}
\]  
(108)
which, in turn, implies that the liquid sheet’s slope at the (removable) singularity is given by
\[
\frac{dH}{dx} = \tan \theta = \frac{-C_{pn} Fr}{\left( We^4 - C_{pn}^2 Fr^2 \right)^{1/2}},
\]  
(109)
where \( We^2 \geq |C_{pn} Fr| \).

The second method for analyzing the singularities is purely algebraic. In this method, Eqs. (76)–(79) are written as
\[
A \frac{dV}{dx} = f,
\]  
(110)
where \( V = (u_0, v_0)^T \) and the components of \( A \) and \( f \) are
\[
an_{11} = 1 - \frac{v_0^2}{We V^3}, \quad a_{12} = a_{21} = \frac{u_0 v_0}{We V^3}, \quad a_{22} = 1 - \frac{u_0^2}{We V^3},
\]  
(111)
\[
f = \left( \frac{1}{u_0 Fr} + \frac{v_0}{u_0 We}, - C_{pn} \frac{1}{We} \right)^T,
\]  
(112)
\( T \) denotes transpose, \( V^2 = u_0^2 + v_0^2 \), and \( h_0 u_0 \) has been set equal to one.

The determinant of \( A \) is nil at the location where \( V = (1/We) \). Therefore, for \( V(0) = 1 \), since \( V \) increases as \( x \) increases on account of the gravitational pull, the singularity occurs downstream from the nozzle exit if \( We < 1 \) and at the nozzle exit if \( We = 1 \). If \( We > 1 \), no singularity occurs.

The matrices that result from the substitution of the first and second columns of \( A \) by \( f \) have zero determinants when
\[
u_0^2 = \frac{1}{We^2} \left( 1 + \left( C_{pn} \frac{Fr}{We^2} \right)^2 \right), \quad v_0 = -C_{pn} \frac{Fr}{We^3}.
\]  
(113)
and, since \( dH/dx = v_0/u_0 \), the slope at the singularity only depends on \( We, Fr \) and \( C_{pn} \). Moreover, \( u_0 \) must be real so that \( We \geq (|C_{pn} Fr|)^{1/2} \), and this result
coincides with the one derived by Finnicum et al. [15] and with that of Eq. (109). At the singularity, \( u_0 \) and \( v_0 \) are given by the above expressions; therefore, the slope of the liquid sheet at the removable singularity is known, if the singularity occurs. This means that, if \( We \leqslant 1 \), the liquid sheet leaves the nozzle at an angle which is not equal to that of the nozzle exit.

Note that both methods, i.e., Taylor series expansions and linear algebra, for analyzing the singularities provide the same result which, in turn, coincides with the one derived by the author for planar liquid membranes [21]. Therefore, despite the assumptions made in the derivation of Eqs. (65) and (76)–(79), these equations are remarkably similar to and have the same singularities as the exact Eqs. (80)–(83) for planar liquid membranes.

7. Presentation of results

In this section, some sample results illustrating the dynamics of planar, liquid sheets are presented. These results have been obtained from the numerical solution of Eqs. (76)–(79) which were discretized by means of finite difference methods which employ first-order discretizations for the convection terms. These time-dependent results were obtained as follows. Eqs. (76)–(79) were first solved iteratively with time-independent boundary conditions at the nozzle exit until a steady state was found, and this state was assigned to \( t = 0^- \) and used as initial condition for the time-dependent studies when either the boundary conditions or the pressure difference across the liquid sheet were varied as functions of time or when mass was injected into the compartments on either side of the sheet. Convergence was defined to occur when

\[
\sum_{i=1}^{\text{NP}} \sum_{j=1}^{4} (\phi_j^{k+1}(t,x_i) - \phi_j^k(t,x_i))^2 \leq 25 \times 10^{-8}
\]

where \( \text{NP} \) denotes the number of grid points in the \( x \)-direction, \( k \) denotes iteration, and \( \phi_j, j = 1, 2, 3 \) and 4, corresponds to \( h, H, u \) and \( v \), respectively.

For the sake of convenience, we introduce the pressure coefficient

\[
C_{pn} = C_p \max \left( \left( -\frac{V_{ss}}{V_e} \right)^{k_e} + \left( \frac{p_e}{p_i} \right)^{ss} \left( \frac{V_{ss}}{V_i} \right)^{k_i} \right) \tag{114}
\]

when there is no mass injection and the gases are assumed to behave polytropically with a polytropic exponent equal to \( k \), and

\[
C_{pn} = C_p \max \left( -\frac{m_e T_{ss} V_{ss}}{m_i T_i V_i} + \frac{m_i T_{ss} V_{ss}}{m_i T_{ss} V_i} \right) \tag{115}
\]

when there is (isothermal) mass injection on either side of the planar liquid sheet, where \( C_p \max = \frac{h_0 p_{ss}^e}{2 \sigma} \), \( m, V \) and \( T \) denote mass, volume and temperature of the gases, respectively, the subscripts e and i denote the gases on the left and right, respectively, of the liquid sheet, and the superscript \( ss \) denotes
reference values; in addition, the reference values used in the nondimensionalization of the governing equations were taken as those as the nozzle exit and the gases are assumed to be ideal. Therefore, the number of dimensionless groups which characterize the dynamics of planar liquid sheets are \( Fr, We, C_p_{\text{max}}, h_0/L, a/L, b/L, \theta_0, \) and \( k_e, k_i \) and \((p_i/p_c)^{\alpha s}\) when there is no mass injection, where

\[
    u(t, 0) = 1, \quad v(t, 0) = \tan \theta_0, \quad h(t, 0) = 1, \quad H(t, 0) = 0 \tag{116}
\]

and \( \theta_0 \) denotes the angle with which the liquid exits the nozzle.

Calculations were first performed with 5001 grid points, \( L = 100 \) and time steps equal to or smaller than \( 10^{-4} \) and \( C_{pm} = KC_{p_{\text{max}}} (1 + A \sin St) \) where \( a \) and \( S \) denote the amplitude and frequency of the imposed pressure difference oscillations and \( K \) is a constant. Since the pressure coefficient was specified, the calculations described briefly in this paragraph correspond to a planar liquid sheet subject to an imposed pressure difference. For \( K = 0.25, A = 0.50, S = 0.1, \theta_0 = 0 \) and \( C_{p_{\text{max}}} = 1, H(L, t) \approx -7.078 \) for \( 50 \leq We \leq 1000 \) and \( 10 \leq Fr \leq \infty \). However, \( H(L, t) \approx -2.496 \) and \(-3.431 \) for \( \theta_0 = 15^\circ \) and \( 30^\circ \), respectively, \( We = 50, Fr = 10, K = 0.25, A = 0.50, S = 0.1, \theta_0 = 0 \) and \( C_{p_{\text{max}}} = 1; H(L, t) \approx -4.312, -0.005 \) and \(-2.823 \) for \( K = 0.50, 1.00 \) and \( 0.10 \), respectively, \( We = 50, Fr = 10, \theta_0 = 0, A = 0.50, S = 0.1 \) and \( C_{p_{\text{max}}} = 1; H(L, t) \approx -2.823 \) and \(-0.005 \) for \( C_{p_{\text{max}}} = 10 \) and \( 100 \), respectively, \( We = 50, Fr = 10, \theta_0 = 0, A = 0.50, S = 0.1 \) and \( K = 0.25 \). \( H(L, t) \) was not found to be very sensitive to \( A \) and \( S \) for \( L = 100, We = 50, Fr = 10, K = 0.25, \theta_0 = 0 \) and \( C_{p_{\text{max}}} = 1; \) for the same values of the parameters, \( H(L, t) \) was found to increase with \( L \). Moreover, the numerical calculations failed when \( We = 1 \) and this is in accord with the analytical results obtained in Section 6.3 for the singularities at the nozzle exit for Weber numbers less than or equal to one, i.e., the time-dependent numerical calculations show a singularity for \( We = 1 \). It must be stated that \( H(L, t) \) was found to vary sinusoidally for the cases investigated; however, its amplitude was much smaller than the mean value of \( H(L, t) \) and, for this reason, the symbol \( \approx \) above refers to mean value.

When the liquid’s axial velocity component at the nozzle exit oscillates in a sinusoidal manner, i.e., \( u(0,t) = 1 + A \sin St, v(0,t) = u(0,t) \tan \theta_0, We = 50, Fr = 10, L = a = b = 100 \) and \((p_i/p_c)^{\alpha s} = 1\), and there is no mass injection of gases into the compartments on either side of the liquid sheet, it has been observed that the planar liquid sheet oscillates, e.g., \( h(L,t) \) and \( H(L,t) \) oscillate, in an almost periodic fashion for \( 1 \leq k_e \leq 1.4, 1 \leq k_i \leq 1.4, \theta_0 = 0^\circ, 15^\circ \) and \( 30^\circ \) degrees, and \( 1 \leq C_{p_{\text{max}}} \leq 25 \) when \( A = 0.25 \) and \( S = 0.1 \). The largest values of \( C_{pm} \) and \( H(L,t) \) increase as \( k_e \) or \( k_i \) are increased from 1 to 1.4, as the Weber number is decreased, and as \( \theta_0 \) and \( C_{p_{\text{max}}} \) are increased.

Fig. 2 shows that neither the pressure coefficient nor the transverse displacement of the liquid sheet are periodic functions of time for \( C_{p_{\text{max}}} = 100 \); in fact, \( R(L) \equiv H(L,t) \) shows peaks of different amplitude, and its phase diagram
occupies a small region and is characteristic of tori; the phase diagram of \( h(L, t) \) is a simple closed curve with a beak shape where the beak corresponds to the smallest values of \( h(L, t) \) which have a speed, \( \partial h/\partial t(L, t) \), close to zero. Although not shown here, there is an initial transient in \( C_p, h(L, t), H(L, t) \) and \( u(L, t) \) when the pressure coefficient evolves from its initial value to much larger pressure oscillations caused by the compression and expansion of the gases on both sides of the liquid sheet. Since the pressure of these gases depends in a nonlinear fashion, for \( k \neq 1 \), on the volume enclosed by the liquid sheet which, in turn, depends on the sheet’s geometry, the pressure of the gases is a strong nonlinear function of the sheet’s geometry and dynamics, and this nonlinear dependence is amplified by \( C_{p_{\max}} \). As stated before, \( H(L, t) \) becomes a periodic function of time for all the values of the parameters investigated provided that \( C_{p_{\max}} \lesssim 25 \).

The nonlinear dynamics of planar liquid sheets was also investigated when \( \theta_0 = B + A \sin St \) which correspond to sinusoidal oscillations of the transverse velocity component at the nozzle exit. For \( We = 50, Fr = 10, L = a = b = 100, C_{p_{\max}} = 1, B = 0, \) no mass injection of gases, and \( (p_i/p_e)^{\alpha} = 1 \), it has been
observed that the planar liquid sheet oscillates, i.e., \( h(L, t) \) and \( H(L, t) \) oscillate, in an almost periodic fashion for \( 1 \leq k_e \leq 1.4 \) and \( 1 \leq k_i \leq 1.4 \) when \( A = 10 \) and \( S = 0.1 \). The largest values of \( C_{pn} \) and \( H(L, t) \) increase as \( k_e \) or \( k_i \) is increased from 1 to 1.4, as the Weber number is decreased, and as \( A \) and \( C_{p_{\text{max}}} \) are increased. For \( C_{p_{\text{max}}} \leq 25 \), it was found that there is an initial transient when the pressure coefficient evolves from its initial value to much larger pressure oscillations caused by the compression and expansion of the gases on both sides of the liquid sheet. For \( We = 50, Fr = 10, L = a = b = 100, C_{p_{\text{max}}} = 1, B = 0, \) no mass injection of gases, \( k_e = k_i = 1, A = 10, S = 0.1 \) and \( (p_i/p_e)^{ss} \neq 1 \), it was observed that \( C_{pn} \) and \( H(L, t) \) became periodic functions of time, whereas \( h(L, t) \) became a periodic function of time with two frequencies as indicated in Fig. 3. Moreover, the amplitude of \( C_{pn} \) increases whereas that of \( H(L, t) \) decreases as \( (p_i/p_e)^{ss} \) is increased; the amplitude of the largest and smallest peaks of \( h(L, t) \) increases and decreases, respectively, as \( (p_i/p_e)^{ss} \) is increased. The axial velocity component at \( x = L \) is also a periodic function of time as indicated in Fig. 3, but the amplitude of the smallest velocity peak

![Fig. 3. Pressure coefficient (top left), \( R(L) \equiv H(L, t) \) (top right), \( b(L) \equiv h(L, t) \) (bottom left) and phase diagram of \( h(L, t) \) (bottom right) for a liquid sheet with \( b_0 = B + A \sin St \). \( B = 0, A = 10, S = 0.1, L = a = b = 100, k_i = k_e = 1, We = 50, Fr = 10, C_{p_{\text{max}}} = 1 \) and \( (p_i/p_e)^{ss} = 1.20; \) no gaseous mass injection.](image-url)
decreases and eventually disappears as \((p_i/p_e)^{ss}\) is increased; in fact, only a velocity peak is observed for \((p_i/p_e)^{ss} = 1.30\).

For \(C_{p_{\text{max}}} = 10\) and 25, the phase diagrams of \(R(L)\) were ellipses whose horizontal semiaxis was about 4, whereas their vertical ones were about 2.5. However, for \(C_{p_{\text{max}}} = 100\), the pressure coefficient exhibited some modulation, and \(h(L,t)\), \(H(L,t)\) and \(u(L,t)\) contained many frequencies and their phase diagrams were characteristic of tori as indicated in Fig. 4. For \(C_{p_{\text{max}}} = 500\), the pressure coefficient showed some bursting characteristics which were analogous to those of \(h(L,t)\), \(H(L,t)\) and \(u(L,t)\), and the phase diagram of the sheet’s transverse displacement and thickness at \(x = L\) showed strange chaotic behaviour as illustrated in Fig. 5.

Calculations were also performed to determine the effects of \(g\)-jitter on the dynamics of planar liquid sheets in the absence of mass injection. Such calculations were performed with

\[
\frac{1}{Fr} = \frac{1}{Fr^{ss}} (1 + A \sin St)
\]

(117)

Fig. 4. Pressure coefficient (top left), \(R(L) \equiv H(L,t)\) (top right), \(b(L) \equiv h(L,t)\) (bottom left) and phase diagram of \(H(L,t)\) (bottom right) for a liquid sheet with \(\theta_0 = B + A \sin St\). \((B = 0, A = 10, S = 0.1, L = a = b = 100, k_i = k_e = 1, We = 50, Fr = 10, C_{p_{\text{max}}} = 100\) and \((p_i/p_e)^{ss} = 1; \text{no gaseous mass injection.})\)
and showed that, for vertically falling liquid sheets, only the longitudinal velocity component and the pressure are periodic functions of time, whereas \( v \) and \( H \) remain zero, thus indicating that the numerical results are accurate.

Only when \( h_0 \neq 0 \) and/or \( p_{ss} \neq p_{ss} \), did \( v \) and \( H \) oscillate as functions of time. However, the amplitudes of the oscillations in \( C_{pn} \), \( u(L,t) \), \( H(L,t) \) and \( h(L,t) \) were found to be small (on the order of \( 10^{-4} \) or smaller) for \( 1 \leq We \leq 1000 \), \( 10 \leq Fr_{ss} < \infty \), \( L = a = b = 100 \), \( 0 \leq \theta_0 \leq 30 \) degrees, \( (p_i/p_e)^{ss} = 1 \), \( 1 \leq k_i \leq 1.4 \), \( 1 \leq C_{p \text{ max}} \leq 10 \), \( 0.25 \leq A \leq 1 \) and \( 0.01 \leq S \leq 0.5 \) which correspond to sinusoidal g-jitter. Only for \( C_{p \text{ max}} = 100 \) did the amplitude of the pressure coefficient reached values close to about 0.04.

Gaseous mass injection was simulated as either

\[
m_i = (m_i)^{ss}(1 + a_i \sin S_i t), \quad m_e = (m_e)^{ss}(1 + a_e \sin S_e t),
\]

or

\[
m_i = (m_i)^{ss}(1 + a_i t) \quad 0 \leq t \leq T_i, \\
m_e = (m_e)^{ss}(1 + a_e t) \quad 0 \leq t \leq T_e.
\]
In either sinusoidal or linear mass injection, it was assumed that the injected gases mixed instantaneously with those on either side of the liquid sheet and the process was assumed to be isothermal. For $We = 50$, $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.05$, $S_i = 0.1$, $a_e = 0$, $S_e = 0$, $C_{p_{\text{max}}} = 1$ and $1 \leq k_e = k_i \leq 1.67$, it has been observed that the planar liquid sheet oscillates, i.e., $h(L, t)$ and $H(L, t)$ oscillate, in a periodic fashion. The largest values of $C_{p_{\text{max}}}$ and $H(L, t)$ were almost independent of $k_e = k_i$ as this parameter was increased from 1 to 1.67; $C_{p_{\text{max}}}$ increases, whereas $H(L, t)$ decreases as the Weber number is decreased; $H(L, t)$ increases as $\theta_0$ is increased; and, $C_{p_{\text{max}}}$ and $H(L, t)$ increase as $C_{p_{\text{max}}}$ is increased. For $We = 50$, $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $C_{p_{\text{max}}} = 1$ and $1 \leq k_e = k_i \leq 1.67$, it has been found that the nonlinear dynamics of planar liquid sheets depends on the amplitudes and frequencies of the mass injection rates on both sides of the sheet, and some sample results are presented in Fig. 6 which indicates that $C_{p_{\text{max}}}$, $H(L, t)$, $u(L, t)$ and $h(L, t)$ are periodic functions of time; the phase diagram of $H(L, t)$ is a simple closed curve which is

![Phase Diagram](image)

Fig. 6. Pressure coefficient (top left), $R(L) \equiv H(L, t)$ (top right), $b(L) \equiv h(L, t)$ (bottom left) and phase diagram of $h(L, t)$ (bottom right) for a liquid sheet with gaseous mass injection. ($\theta_0 = 0$, $L = a = b = 100$, $k_i = k_e = 1$, $We = 50$, $Fr = 10$, $C_{p_{\text{max}}} = 1$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.05$, $a_e = 0.025$, $S_i = 0.1$ and $S_e = 0.025$.)
distorted as either $a_i$ or $a_e$ is increased, whereas that of $h(L,t)$ exhibits a number of loops which depends on the amplitude and frequency of the sinusoidal mass injection rate as indicated in Table 1.

Complex dynamic behaviour was observed for $H(L,t)$, $h(L,t)$ and $u(L,t)$ (but not for $C_p$) for $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.05$, $S_i = 0.1$, $a_e = 0$, $S_e = 0$, $C_p \text{ max} = 1$ and $k_e = k_i = 1$ as the Weber number was varied. For example, the number of loops in the phase diagram of $h(L,t)$ was 4, 2, 1, 1, 1 and 1 for $We = 1, 1.5, 2.5, 5, 10$ and 15, respectively; the phase diagrams of $h(L,t)$ became more elliptical as the Weber number was increased and those of $H(L,t)$ only showed a single loop and also became more elliptic as the Weber number was increased.

For $We = 50$, $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.005$, $S_i = 0.1$, $a_e = 0$, $S_e = 0$ and $k_e = k_i = 1$, bifurcation phenomena were observed as $C_p \text{ max}$ was increased. For $C_p \text{ max} \leq 25$, $H(L,t)$, $h(L,t)$ and $u(L,t)$ were found to be periodic functions of time, and the phase diagrams of both $h(L,t)$ and $H(L,t)$ had only a single loop and became more elliptically shaped as $C_p \text{ max}$ was increased; however, for $C_p \text{ max} = 100$, the nonlinear integro-differential coupling between the pressure and volume of the gases on either side of the liquid sheet and the fluid dynamics of the liquid resulted in complex nonlinear behaviour such as the one illustrated in Fig. 7. Similar complex behaviour was also observed for $We = 50$, $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.05$, $S_i = 0.1$, $a_e = 0$, $S_e = 0$ and $k_e = k_i = 1$, but for lower values of $C_p \text{ max}$ than for $a_i = 0.005$, and this result implies that the nonlinear dynamics of planar liquid sheets is a strong function of $C_p \text{ max}$ and the amplitude of the

### Table 1

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<th>$a_i$</th>
<th>$a_e$</th>
<th>$S_i$</th>
<th>$S_e$</th>
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Calculations have been also performed with linear mass injection in order to determine the effects of the mass injection rate and duration (cf. Eq. (119)). The results of these calculations indicate that the pressure coefficient increases almost linearly until $t = T_i$, i.e., the end of injection, and then decreases and oscillates in a damped manner until a new steady state corresponding to the (new) higher mass on either side of the liquid sheet. A similar behaviour has also been observed in $H(L, t)$ and $u(L, t)$, whereas $h(L, t)$ first decreased and then increased quite rapidly before decreasing in a damped manner to its final steady state value. The largest value of $C_{pm}$ was found to be independent of $1 \leq k_c = k_i \leq 1.67$ for $We = 50$, $Fr = 10$, $L = a = b = 100$, $\theta_0 = 0$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.005$, $a_e = 0$, $S_i = 0.1$ and $S_e = 0.$

periodic mass injection rate when there is gaseous mass injection on either side of the liquid sheet.

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Fig. 7. Pressure coefficient (top left), $R(L) \equiv H(L, t)$ (top right), $h(L) \equiv h(L, t)$ (bottom left) and phase diagram of $H(L, t)$ (bottom right) for a liquid sheet with gaseous mass injection. ($\theta_0 = 0$, $L = a = b = 100$, $k_i = k_c = 1$, $We = 50$, $Fr = 10$, $C_{p \text{ max}} = 1$, $(p_i/p_e)^{ss} = 1$, $a_i = 0.005$, $a_e = 0$, $S_i = 0.1$ and $S_e = 0.$)
were found to become periodic functions of time long after the end of the linear mass injection, whereas, for $C_p^{\text{max}} = 1$, the number and duration of the damped oscillations of $C_{pn}$ were found to increase as the Weber number was decreased towards one.

The results discussed in this section and others not shown here indicate that the nonlinear dynamics of confined, inviscid, planar liquid sheets and membranes is a strong function of the amplitude and frequency of the pressure forcing and injection rates, and that, due to the nonlinear integro-differential coupling between the liquid and the gases that surround it, bifurcation phenomena can occur as the Weber number is decreased or the pressure coefficient is increased. These phenomena may be characterized by multiple frequencies related rationally or tori. Moreover, in some cases, the compressibility of the gases that surround the liquid can result in periodic transverse oscillations of the liquid sheet even after the injection of gases is ended.

A comparison between the results presented in Figs. 4–7 and those of Casperson [11,12] clearly indicate that the compressibility of the gases on both sides of the liquid sheet introduce more nonlinearities than those associated with fluttering fountains where either $p_i$ or $p_e$ are constant.

8. Conclusions

One-dimensional models of planar liquid sheets surrounded by dynamically passive gases have been developed by means of perturbation methods based on the slenderness ratio, integral formulations based on the integration of the Euler equations across the sheet, variational techniques based on moments of the conservation equations, and series expansions. By means of perturbation methods, one-dimensional equations for inviscid, slender and thin planar liquid sheets have been obtained and two different flow regimes corresponding to large and small Weber numbers have been identified. Inertia effects control the large Weber number regime where the longitudinal velocity component depends on gravity, whereas a third-order differential equation results for small Weber numbers; this third-order equation has analytical solutions in terms of elliptic integrals.

It has been shown that methods based on the expansion of the liquid velocity components and pressure about the sheet’s midline result in identical models to those based on integral formulations which assume that the liquid’s longitudinal velocity component is uniform across the liquid sheet and the pressure is a linear function of the transverse coordinate; they are also identical to those based on variational formulations which assume that the longitudinal velocity component is uniform whereas the transverse one is a linear function of the transverse coordinate. Approximate analytical solutions to some model equations derived from integral formulations have been obtained as functions
of the Froude and Weber numbers and pressure difference across the liquid sheet for steady state flows. The asymptotic equivalence between models obtained from integral formulations and perturbation methods has been established for thin and slender liquid sheets.

It has also been shown that the steady equations obtained from integral formulations exhibit singularities at or below the nozzle exit for Weber numbers equal to or less than 1, respectively. These singularities are removable and have been analyzed by means of algebraic and differential techniques which provide the same results.

The nonlinear dynamics of inviscid, planar liquid sheets surrounded by dynamically passive gases has also been studied numerically by means of a finite difference technique as a function of the nondimensional parameters that control the fluid dynamics of these sheets when they are forced by imposed pressure fields, time-dependent boundary conditions, and injection of gases into the volumes that surround the liquid. It has been found that the pressure of the gases depends in a nonlinear manner on the geometry and transverse displacement of the sheet and may result in rich dynamic behaviour at low Weber numbers or when the pressure coefficient is sufficiently large. For example, it has been observed that the number of loops exhibited by the phase diagram for the sheet's thickness evolves from one to four as the frequency of the sinusoidal mass injection is decreased. It has also been observed that the phase diagram may exhibit characteristics which are typical of tori.

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