Complex patterns in three-dimensional excitable media with advection

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Abstract

Wave propagation in three-dimensional excitable media subject to homogeneous Neumann boundary conditions and subject to irrotational or rotational advective fields which satisfy the no-penetration condition is studied numerically. It is shown that the velocity field annihilates spiral waves and results in complex periodic spatio-temporal patterns which are characterized by either planar or curved fronts depending on the flow compressibility, number and location of stagnation points, rotation and straining. For sinusoidal velocity fields with a frequency equal to one, almost planar fronts which resemble those found in two-dimensional excitable media are found, whereas, when the frequency is increased, several curved fronts propagate at about the same speed and surround each other when they approach an edge of the domain. It is also shown that the main effect of increasing the frequency of the velocity field on the concentrations at fixed monitor locations is to decrease the period of the spikes of the activator’s concentration.

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1. Introduction

The stability of moving fronts in two-dimensional excitable media depends on the convective and diffusive transport, because transport provides the coupling between spatially separated elements and brings about the front propagation [1]. Of the two kinds of transport, the diffusive one has been more extensively studied than the convective transport, especially in biological media [2].

Biktashev et al. [3,4] considered an excitable medium in two dimensions with a cubic non-linearity given by the FitzHugh–Nagumo system and a shear characterized by a velocity field in the $x$-direction which is either a linear or a sinusoidal function of the $y$-coordinate, and showed that the shear can distort and then break spiral waves. Such breaks were found to result in a chain reaction of spiral wave births and deaths. The velocity fields employed by Biktashev et al. [3] are one-directional and solenoidal, but not irrotational, and they do not satisfy the no-penetration condition at the boundaries of the domain; in fact, these authors used periodicity conditions for the sinusoidal velocity field.

Convective transport was also studied by Wellner et al. [5] who considered the drift of stable, meandering spiral waves in a singly diffusive two-dimensional FitzHugh–Nagumo medium caused by a weak time-independent gradient or convection in the fast-variable equation, showed by means of perturbation methods the equivalence between gradient and convective perturbations, and proposed a semiempirical solution to the drift of two-dimensional spiral waves that depends on the period of rotation and the value of the fast variable at the center of the spiral wave.
Kærn and Menzinger [6] considered a one-dimensional reaction-diffusion equation and a plug flow velocity, and predicted the existence of stationary waves quite different from those associated with the Turing mechanism which requires a fast inhibitor diffusion for the formation of spatially periodic patterns. The stationary waves obtained by Kærn and Menzinger [6] were also found to be quite different from those associated with differential flow instabilities. Andresén et al. [7] considered the formation of stationary periodic patterns in a one-dimensional excitable medium with a constant plug flow and the Brusselator model, and showed that their model may exhibit such patterns even in the case of equal diffusion coefficients for certain types of boundary conditions in an open system.

More recently, Ramos [8] has studied the propagation of spiral waves in two-dimensional solenoidal velocity fields which are not irrotational but satisfy the no-slip boundary conditions, and shown that the advective field distorts the spiral wave at moderate frequencies, whereas, at large frequencies, the average shape of the spiral wave is nearly identical to that in the absence of convection, although its inner and outer parts exhibit spatial oscillations whose frequency increases as the frequency of the velocity field is increased. At low frequencies and high amplitudes of the velocity field, the concentration of the activator and the wave propagation are controlled by the symmetry of the velocity and the number and location of the stagnation points, and the concentration of the activator may exhibit either counter-rotating regions or a layered structure. Ramos [8] also studied the convection-induced anisotropy in solenoidal velocity fields by determining the equivalent anisotropic diffusive tensor which, when employed in excitable media governed by reaction-diffusion equations, would result in the same patterns as those observed in two-dimensional excitable media with flow straining. He has also shown that solenoidal flow fields introduce anisotropic effects which may be quite different from those observed in an anisotropic reactive-diffusive medium [9].

Ramos [10] has also studied the dynamics of spiral waves in two-dimensional non-solenoidal velocity fields which are not irrotational but satisfy the no-slip boundary conditions, and shown that the advective flow field may break up spiral waves for flow speeds on the order of one or form curved fronts whose thickness is on the order of the size of the domain when the flow speed is larger than one. On the other hand, numerical studies of wave propagation in two-dimensional excitatory media subject to non-solenoidal advective flow fields which satisfy the no-penetration condition on the boundaries of the domain indicate that, depending on the amplitude and spatial frequencies of the velocity field, the spiral wave may be distorted highly, break up into a number of smaller spiral waves, or exhibit polygonal shapes or tile patterns. These patterns reflect the symmetry/asymmetry of the velocity field and are characterized by thick regions of high activator's concentration at stagnation points where the velocity gradient is largest, and thin ones which are parallel to the velocity vector [11]. These two-dimensional numerical studies also indicate that the advective field distorts the spiral wave by decreasing its thickness where the velocity is largest due to the stretching of the wave, and by increasing it at the stagnation points where the curvature of the wave is largest.

The break-up of spiral waves as well as the formation of thick fronts and the formation of tiles in two-dimensional excitatory media have been explained in terms of the straining of the wave front and convection-induced anisotropy in non-solenoidal velocity fields [8,10,11] in an analogous manner to that employed in combustion theory to explain flame extinction [1]. Moreover, the boundary conditions on the velocity field also play a role in determining the spiral waves dynamics and its break-up. When the velocity field satisfies the no-slip condition, the boundaries of the domain are stagnation lines where convective effects are nil, whereas the velocity field is parallel to the boundaries when the no-penetration condition is employed. In addition, the velocity fields employed in previous two-dimensional numerical studies were sinusoidal and contained several stagnation points when the spatial frequency was high [8,10,11]; therefore, complex patterns were observed at high spatial frequencies due to both the large number of stagnation points and the straining of the reaction fronts at these points.

Most of the numerical studies on the effects of convection in excitable media performed to date have been two-dimensional [3,8,10,11], and have attributed the spiral wave's break-up to the anisotropy introduced by the velocity field, flow strain and rotation. In this paper, we present a three-dimensional numerical study of the effects of advective velocity fields on excitable media. The study is based on the two-equation Oregonator model of the Belousov–Zhabotinsky (BN) system and employs a three-dimensional, stationary, convective/advective flow field with straining and rotation. The velocity components are sinusoidal functions of the Cartesian coordinates \( x, y \) and \( z \), and are assumed to be the same for both the fast and slow variables, i.e., the same for both the activator and the inhibitor. The main objective of the study is to determine the effects of the flow field and its frequency on wave propagation in three-dimensional excitatory media.

2. Governing equations

The three-dimensional numerical study presented here is based on the BZ reaction which is often modelled by the two-equation Oregonator model [12,13] and may be written as
where \( t \) is time, \( u \) and \( v \) denote the concentrations of the activator and the inhibitor, respectively, \( d_u \) and \( d_v \) are the (constant) diffusion coefficients for \( u \) and \( v \), respectively, \( \mathbf{v} = (v_x, v_y, v_z) \) is the velocity field where the subscripts \( x, y \) and \( z \) denote the corresponding axes, and the source terms in Eqs. (1) and (2) can be written as

\[
F_u = \frac{1}{\epsilon} \left( u - u^2 - fx - q \right), \quad F_v = u - v,
\]

where \( \epsilon = 0.01, f = 1.4 \) and \( q = 0.002 \) and are the same as those employed in the BZ model.

In this paper, it is assumed that \( d_u = 1 \) and \( d_v = 0.6 \), and it is known that, for these values, the Oregonator model has spiral wave solutions in two-dimensional excitable when \( \mathbf{v} = 0 \) and homogeneous Neumann boundary conditions are imposed on all the boundaries of the domain.

Eqs. (1) and (2) were solved in the spatial domain \( \Omega = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2] \times [-L_z/2, L_z/2] \) where \( L_x = L_y = L_z = 15 \) subject to homogeneous Neumann boundary conditions on all the boundaries and to the following initial condition

\[
\begin{align*}
\mathbf{u}(0, x, y, z) &= 0 \quad \text{for } 0 < t < 0.5; \\
\mathbf{v}(0, x, y, z) &= q(f + 1)/(f - 1) \quad \text{elsewhere,}
\end{align*}
\]

unless stated otherwise, where \( \tan \theta(0, x, y, z) = y/x \) is the angle with respect to the \( xy \)-plane measured counter-clockwise from the positive \( x \)-axis. In the absence of convection and in two-dimensional excitable media, this initial condition results in the formation of a spiral wave which rotates counter-clockwise.

The velocity field employed in this study can be written as

\[
\begin{align*}
v_x &= A \cos \left( \frac{m_x \pi x}{L_x} \right) \sin \left( \frac{n_y \pi y}{L_y} \right) \sin \left( \frac{p_z \pi z}{L_z} \right), \\
v_y &= A \sin \left( \frac{m_x \pi x}{L_x} \right) \cos \left( \frac{n_y \pi y}{L_y} \right) \sin \left( \frac{p_z \pi z}{L_z} \right), \\
v_z &= A \sin \left( \frac{m_x \pi x}{L_x} \right) \sin \left( \frac{n_y \pi y}{L_y} \right) \cos \left( \frac{p_z \pi z}{L_z} \right),
\end{align*}
\]

where \( A \) is a constant which has been set equal to 3, and \( m_x, m_y, m_z, n_x, n_y, n_z, p_x, p_y, p_z \) and \( p_z \) are integers; this velocity field satisfies the no-slip condition at the boundaries if \( m_x, n_x, \) and \( p_z \) are odd integers and \( n_y, p_y, m_z, p_z, n_z, m_z, \) and \( n_z \) are even integers, and it does satisfy the no-penetration condition if \( m_x, m_y, m_z, n_x, n_y, n_z, p_x, p_y, \) and \( p_z \) are odd integers.

In this paper, we assume that \( m_x = m_y = m_z = m, n_x = n_y = n_z = n \) and \( p_x = p_y = p_z = p \), and \( m, n \) and \( p \) are odd integers; therefore, the velocity field employed in this paper satisfies the no-penetration condition at the boundaries, i.e., \( \mathbf{v} \cdot \mathbf{n} = 0 \) and \( \mathbf{v} \cdot \mathbf{t} \neq 0 \), where \( \mathbf{n} \) and \( \mathbf{t} \) are the unit vectors normal and tangential, respectively, to the boundaries. Moreover, since \( L_x = L_y = L_z = L \), it can be easily shown that

\[
\nabla \cdot \mathbf{v} = -\pi \left( \frac{m}{L_x} + \frac{n}{L_y} + \frac{p}{L_z} \right) \sin \left( \frac{m \pi x}{L_x} \right) \sin \left( \frac{n \pi y}{L_y} \right) \sin \left( \frac{p \pi z}{L_z} \right)
\]

is different from zero, and the velocity field is not solenoidal, introduces anisotropy due to the spatial dependence of the velocity components, is compressible and affects the effective reaction rate; in fact, it decreases the reaction rate if \( \nabla \cdot \mathbf{v} > 0 \) and decreases it, otherwise.

In addition, the flow field considered in this paper, is irrotational, i.e., \( \omega = \nabla \times \mathbf{v} = 0 \), if \( m = n = p \). The strain rate tensor, \( \mathbf{S} \), is characterized by large stretching, i.e., large \( S_{xx} = \partial v_x/\partial x, S_{yy} = \partial v_y/\partial y, \) and \( S_{zz} = \partial v_z/\partial z, \) on the boundaries; \( S_{xy}, S_{xz}, \) and \( S_{yz} \) are nil on \( x = \pm(L_x/2), y = \pm(L_y/2) \) and \( z = 0, x = \pm(L_x/2), \) \( y = 0 \) and \( z = \pm(L_z/2), \) and \( x = 0, y = \pm(L_y/2) \) and \( z = \pm(L_z/2), \) respectively.

Eqs. (1) and (2) were solved numerical by means of an implicit, time-linearized, second-order accurate (in both space and time) finite difference method [14], and the resulting linear algebraic equations at each time level were solved by means of a conjugate gradient technique. Computations were performed on a 201 \( \times \) 201 \( \times \) 201 point equally spaced mesh.
and a time step of $4 \times 10^{-4}$, unless stated otherwise, in a multiprocessor environment that makes use of parallelization techniques [15]; computations were also performed with equally spaced meshes of $101 \times 101 \times 101$ equally spaced points and a time step of $4 \times 10^{-4}$ in order to insure that the results were independent of the number of grid points.

3. Presentation of results

In this section, some sample results which show wave propagation in three-dimensional excitable media subject to the velocity field given by Eqs. (6)–(8) are presented. Since three-dimensional computations are very demanding, we have obtained results for $m = n = p = 1$, and $m = p = 1$ and $n = 3$. For $m = n = p = 1$, the isosurfaces of $u = 0.90u_{\text{max}}$ where $u_{\text{max}}(t) = \sup_{(x,y,z) \in \Omega} [u(x,y,z,t)]$ for $(x,y,z) \in \Omega$, shown in Fig. 1 indicate that no spiral wave is present; instead, the isosurfaces of the activator concentration indicate that an almost flat front propagates from the corner located at $(x,y,z) = (-L_x/2,-L_y/2,-L_z/2)$ towards that at $(L_x/2,L_y/2,L_z/2)$, although it presents some curvature at the edges $(-L_x/2,-L_y/2,L_z/2)$, $(x,-L_y/2,-L_z/2)$ and $(-L_x/2,y,-L_z/2)$ and may recede a little towards the corner at $(x,y,z) = (-L_x/2,-L_y/2,-L_z/2)$ as a comparison between the first and second frames clearly shows, before it propagates again towards the corner at $(L_x/2,L_y/2,L_z/2)$. When this front is about to reach the boundary points at $(0,-L_y/2,-L_z/2)$, $(-L_x/2,0,-L_z/2)$ and $(-L_x/2,-L_y/2,0)$, a second parallel front appears between the first front and the corner located at $(-L_x/2,-L_y/2,-L_z/2)$ (cf. the third frame of Fig. 1), and both fronts propagate towards the

![Fig. 1. Isosurfaces of $u = 0.90u_{\text{max}}$ where $u_{\text{max}}(t) = \sup_{(x,y,z) \in \Omega} [u(x,y,z,t)]$ for $(x,y,z) \in \Omega$ at (from left to right, from top to bottom) $t = 94.440, 94.500, 94.644, 94.724, 94.740, 94.764, 94.798, 94.948, 94.996$ and $95.076$ for $m = n = p = 1$.](image-url)
corner at \((L_x/2, L_y/2, L_z/2)\). A little after the first front passes the points \((-L_x/2, -L_y/2, L_z/2)\), \((L_x/2, -L_y/2, 0)\) and \((-L_x/2, L_y/2, 0)\), this front develops a protuberance or peak towards the corner (cf. the fourth frame of Fig. 1) at \((L_x/2, L_y/2, L_z/2)\) which, when it reaches this corner becomes another almost flat front which together with the corner located at \((L_x/2, -L_y/2, L_z/2)\) forms a pyramid (cf. the fifth frame of Fig. 1). This front propagates towards the apex of this pyramid and then disappears from the domain, whereas the second front continues its travel towards the corner at \((L_x/2, L_y/2, L_z/2)\) and then acquires a pyramidal shape almost identical to that of the first front. The second front also propagates towards the apex of the pyramid (cf. the ninth frame of Fig. 1) and then disappears from the domain.

The isosurfaces of \(v = 0.90v_{\text{max}}\) where \(v_{\text{max}}(t) = \sup \{v(x,y,z,t)| (x,y,z) \in \Omega\}\) for \((x,y,z) \in \Omega\) are presented in Fig. 2 which indicates that an almost planar front emerges from the back’s lower left corner and propagates towards the front’s upper right corner; this front recedes a little towards the back’s lower left corner and then propagates once again towards the front’s upper right corner. Some time after this planar front passes the front’s lower left and the back’s lower right and upper left corners, a protuberance is developed, and, once this protuberance reaches the right boundary, a second planar front emerges near the back’s lower left corner, whereas the first front acquires an almost flat shape and propagates towards the front’s upper left corner, from which it disappears. The second front follows a similar behaviour to the first one,
although it remains almost stationary at the front’s lower left and the back’s upper left and lower right corners for quite a long time.

It should be noted that the frames presented in Figs. 1 and 2 correspond to times when \( u \) and \( v \), respectively, show activity. As it will be discussed below, \( u \) is characterized by spikes whose width is much smaller than the period of \( v \); therefore, the times at which \( u \) and \( v \) are presented in Figs. 1 and 2, respectively, do not coincide. It should be also noted that \( u_{\text{max}} \) and \( v_{\text{max}} \) are time-dependent, and the distance between the first and second fronts discussed above indicates the thickness of the wave. In fact, if the values of \( u \) and \( v \) are drawn throughout the domain, one can see that a thick wave front characterizes the third and fourth frames and the left part of the fifth frame of Fig. 1, whereas the wave’s thickness is small in the other frames of this figure.

The values of \( u \) and \( v \) at the monitor locations \( (x_m,y_m,z_m) = (-L_z/2,-L_y/2,-L_x/2) + (x_m,y_m,z_m) \) where \( (x_m,y_m,z_m) = (25\delta,25\delta,25\delta), (13\delta,13\delta,13\delta), (37\delta,37\delta,37\delta), (13\delta,37\delta,13\delta), (37\delta,37\delta,13\delta), (13\delta,13\delta,37\delta), (37\delta,13\delta,37\delta), \) and \( (13\delta,37\delta,37\delta); \) where \( \delta = L_x/100 \), indicate that the activator’s concentration raises sharply to a maximum value equal to 0.9248 and then decreases to a very small value. The period or distance between two successive peaks and the width of \( u \) are 4.4440 and 0.3640, respectively, regardless of the monitor point; the width of \( u \) was measured using the threshold of 0.0040. The maximum and minimum values of \( u \) at the monitor locations were 0.2114 and 0.0079, and the value of \( v \) at the monitor location was characterized by a rapid rise to its maximum value and a slow decrease to its minimum one. The behaviour of \( v \) at the monitor locations resembles that observed in relaxation oscillations, whereas that of \( u \) indicates a spiky activity.

For \( m = p = 1 \) and \( n = 3 \), the isosurfaces of \( u(x,y,z,t) \leq 0.72 \) shown in Figs. 3 and 4 indicate that a curved front emerges from a region at the left and top boundaries which expands towards the right, front, back and bottom boundaries while increasing its perimeter on the left and top boundaries. When this front reaches the front boundary, a second front emanates from almost the same region as the first one and both fronts propagate parallel to each other at nearly constant speed; the distance between these fronts provides an indication of the wave’s thickness. When the first front reaches the right, bottom and front boundaries, its shape folds on these planes and the resulting pattern moves almost radially towards the edge \( (x,L_x/2,0) \). The second front exhibits a similar dynamical behaviour as the first one and, eventually, surrounds and becomes almost concentric with it along the previously mentioned edge. The resulting pattern resembles a sausage whose largest and smallest cross-sections are at the back and, respectively, boundaries, and exhibits a contraction at \( (0,L_x/2,0) \). The parts of the sausage-like front between the contraction point and the front and back boundaries move towards the back boundary, while their cross-sections decrease; eventually, the contraction cross-section becomes nil and two separate fronts appear. The front whose largest cross-section is on the back boundary exhibits a conical shape, whereas that which is located between the front boundary and the contraction, resembles a droplet whose size decreases to zero long before the conical front’s apex moves towards the corner located at \( (-L_z/2,L_y/2,0) \), and then disappears from the domain.

For \( m = p = 1 \) and \( n = 3 \), the values of \( u \) and \( v \) at the monitor locations mentioned above exhibit similar trends to those for \( m = n = p = 1 \), except that the maximum value of \( u \) is equal to 0.9245 and then decreases to a very small value, and the period or distance between two successive peaks and the width of \( u \) are 3.9996 and 0.3560, respectively, regardless of the monitor point; the width of \( u \) was measured using the threshold of 0.0040. The maximum and minimum values of \( v \) at the monitor locations were 0.2112 and 0.0079, and the value of \( v \) at the monitor locations was characterized by a rapid rise to its maximum value and a slow decrease to its minimum one. Therefore, the main effect of an increase of \( n \) on the time histories of \( u \) and \( v \) is to decrease the period of the peaks in the activator’s concentration. However, the results discussed previously indicate that the advective flow field annihilates spiral waves and yields complex spatio-temporal patterns characterized by planar or curved fronts.

The isosurfaces of the activator’s concentration for \( m = p = 1 \) and \( n = 3 \) exhibit similar trends to those of the inhibitor’s concentration, although, as indicated previously, \( u \) has a spiky structure of much lower duration than that of \( v \). As a consequence, \( v \) is above its lowest value for longer time than \( u \).

The differences between the results corresponding to \( m = n = p = 1 \), and \( m = p = 1 \) and \( n = 3 \) can be explained as follows. First, since the flow fields employed in this study are not solenoidal, compressibility affects the effective reaction rate by introducing the terms \( -u\nabla \cdot v \) and \( -v\nabla \cdot v \) in Eqs. (1) and (2), respectively, or if the convection terms are written in strong-conservation law form, i.e., as \( \nabla \cdot (uv) \) and \( \nabla \cdot (vx) \) in Eqs. (1) and (2), respectively. Second, for \( m = n = p = 1 \), \( u_x = u_y = u_z = 0 \) at \( x = \pm(L_x/2), y = 0 \) or \( z = 0 \), \( v_x = v_y = v_z = 0 \) at \( x = 0, y = \pm(L_y/2) \) or \( z = 0 \), \( u_x = u_y = u_z = 0 \); therefore, the only stagnation point occurs at \( x = y = z = 0 \). Moreover, the compressibility of the velocity field is nil at \( (x,y,z) = (0,0,0) \) and largest at \( (x,y,z) = (L_x/2,L_y/2,L_z/2) \); therefore, the contribution of the flow compressibility to the effective reaction rate is nil at \( (x,y,z) = (0,0,0) \), and this explains the protuberance or peak that is observed when the almost planar fronts corresponding to \( m = n = p = 1 \) reach this point. However, for \( m = p = 1 \) and \( n = 3 \), the compressibility of the velocity field is nil at \( (x,y,z) = (0,0,0) \) and smallest at \( (x,y,z) = (L_x/2,L_y/2,L_z/2) \), i.e., there is flow expansion at \( (x,y,z) = (L_x/2,L_y/2,L_z/2) \), and this expansion reduces the effective reaction rate at this
point. The absolute value of the compressibility of the velocity field corresponding to \( m = p = 1 \) and \( n = 3 \) is \( \frac{5}{3} \) larger than that corresponding to \( m = n = p = 1 \).

For \( m = p = 1 \) and \( n = 3 \), \( v_x = 0 \) at \( x = \pm(L_x/2) \), \( y = 0 \), \( y = \pm(L_y/3) \) or \( z = 0 \), \( v_y = 0 \) at \( x = 0 \), \( y = \pm(L_y/6) \), \( y = \pm(L_y/2) \) or \( z = 0 \), \( v_z = 0 \) at \( x = 0 \), \( y = \pm(L_y/3) \) or \( z = \pm(L_z/2) \), and this flow has only one stagnation point at \((x, y, z) = (0, 0, 0)\). Furthermore, \( S_{xy} = \pm A \) at \((0, 0, \pm(L_x/2))\), \( S_{xz} = \pm A \) at \((0, \pm(L_y/2), 0)\) and \( S_{yz} = \pm A \) at \((\pm(L_x/2), 0, 0)\) for \( m = n = p = 1 \), whereas \( S_{xy} = \pm 2A \) at \((0, 0, \pm(L_x/2))\) and \((0, \pm(L_y/3), 0)\), \( S_{xz} = \pm A \) at \((0, \pm(L_y/6), 0)\) and \( S_{yz} = \pm 2A \) at \((\pm(L_x/2), 0, 0)\) and \((\pm(L_x/2), \pm(L_y/3), 0)\) for \( m = p = 1 \) and \( n = 3 \); therefore, the straining of the reaction front caused by the velocity field increases as \( n \) is increased and this explains the curvature of and the interaction between the fronts and the boundaries as well as the folding and stretching of the conical fronts discussed above.

As indicated previously [8,10,11], the motion in the vicinity of a point in a three-dimensional excitable medium can be resolved into a uniform translation \( \mathbf{v} \), a rigid body rotation with angular velocity equal to \( \frac{1}{2} \nabla \times \mathbf{v} \) and a pure straining motion. The first two do not have any effect on the internal structure of a locally planar wave front, whereas the third one is described by the symmetric strain rate tensor \( \mathbf{e} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \), where the superscript T denotes transpose. For locally planar wave fronts, the most important decomposition of \( \mathbf{e} \) at stagnation points is associated with \( b = \mathbf{n} \cdot \mathbf{e} \cdot \mathbf{n} \) where \( \mathbf{n} \) is the unit vector normal to the front. In a three-dimensional stagnation point flow, the rate of change of the transverse component of velocity with transverse distance is \( \mathbf{t} \cdot \mathbf{e} \cdot \mathbf{t} \) where \( \mathbf{t} \) is the unit vector tangent to the front and this is equal to \( -b \). This means that a decrease in the normal mass flux with distance through the front, i.e., \( b < 0 \), is
reflected in a net transverse outflow, i.e., $\mathbf{t} \cdot \mathbf{e} \cdot \mathbf{t} > 0$. In combustion theory, this outflow is referred to flame stretch [1], and the stretching can be written as $-\mathbf{n} \cdot \nabla \times (\mathbf{v} \times \mathbf{n}) = -\mathbf{n} \cdot \nabla)(\mathbf{v} \cdot \mathbf{n}) + \nabla \cdot \mathbf{v}$. For the non-solenoidal velocity field employed in this paper, the last term of this expression is not nil.

The results presented in this paper indicate that advective flow fields may destroy spiral waves and result in either planar or curved fronts in three-dimensional excitable media, and that an increase in the spatial frequency of the velocity field results in a decrease of the period of the resulting wave phenomena, although the largest values of $u$ and $v$ and the width of $u$ are nearly insensitive to an increase in the flow frequency. The appearance of either planar or curved fronts in three-dimensional excitable media is analogous to that observed in two-dimensional ones where it has been found that the amplitude, frequency, straining, compressibility and angular speed of and the boundary conditions on the velocity field play a paramount role in determining the break-up or annihilation of spiral waves in excitable media [8,10,11].

A comparison between the results presented here and the two-dimensional ones previously reported by one of the authors [8,10,11] who also used sinusoidal velocity fields with multiple stagnation points indicates that both flow straining and flow rotation play a paramount role in determining wave propagation in two- and three-dimensional excitable media; flow straining distorts the reaction front and aligns it with one of the principal directions of the deformation rate tensor near stagnation points, whereas advection may result in almost uniform composition if it is sufficiently large.

Fig. 4. Isosurfaces of $u = 0.72$ at (from left to right, from top to bottom) $t = 93.600, 93.624, 93.648, 93.680, 93.704, 93.728, 93.912, 93.936$ and $94.016$ for $m = p = 1$ and $n = 3$. 
4. Conclusions

The effects of stationary, advective flow fields which satisfy the no-penetration condition at the boundaries of the domain, on wave propagation in three-dimensional excitable media have been investigated numerically as functions of the spatial frequency of the flow. It has been shown that the advective field does not allow for the propagation of spiral waves; instead, almost planar or curved fronts and complex patterns are observed. These fronts may advance, then recede, and later propagate towards a corner of the domain, and change direction upon reaching a boundary when the spatial frequencies in the three coordinate directions are equal to one. These fronts are initially very thin, but their thickness increases and the fronts develop a protuberance as they reach the center of the domain.

Complex periodic spatio-temporal patterns are observed when the spatial frequency of the velocity field is increased, i.e., as the number of zeroes of one of the velocity components is increased. These patterns may be initially thin fronts which become thick ones upon reaching the center of the domain, and then form folds and wrap around one of the domain edges giving rise to sausage-like patterns. These patterns move towards the boundaries of the domain and acquire conical shapes before disappearing from the excitable medium.

It has also been found that the main effect of increasing the spatial frequency of the flow field in three-dimensional excitable media on the time histories of both the activator’s and the inhibitor’s concentrations at fixed monitor locations is to reduce the period of the activator’s spikes, whereas the maximum and minimum concentrations and the width of the activator’s spikes are nearly independent of such a frequency.

The appearance of almost planar fronts in three-dimensional excitable media subject to stationary flow fields has been explained in terms of the anisotropy, straining, rotation, compressibility and the number of stagnation points introduced by the velocity field. The compressible effects reduce the effective reaction rate if the flow expands, whereas they increase it, if the divergence of the velocity field is negative. On the other hand, the rate of strain tensor results in stretching of the reaction front. Both the destruction and the appearance of almost planar fronts in three-dimensional excitable media are analogous to those found in two-dimensional ones, and indicate that flow rotation and straining play a paramount role on determining wave propagation in excitable media.

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