Set Functors, $L$-Fuzzy Set Categories, and Generalized Terms

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Abstract—In this work, we generalize previous constructions of fuzzy set categories, introduced in [1], by considering $L$-fuzzy sets in which the values of the characteristic functions run on a completely distributive lattice, rather than in the unit real interval. Later, these $L$-fuzzy sets are used to define the $L$-fuzzy categories, which are proven to be rational. In the final part of the paper, the $L$ fuzzy functors given by the extension principles are provided with a structure of monad which is used, together with the functorial definition of the term monad, to provide monad compositions as a basis for a notion of generalised terms. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Fuzziness is more the rule than the exception in practical problems, for usually there is no well-defined best solution for a given problem. Fuzzy set theory is based on the idea that many nonmathematical properties cannot be adequately described in terms of crisp sets comprising those elements that fulfill a given property. Therefore, the notion of membership is considered as a gradual property for fuzzy sets.

A lot of research is being done on fuzzy sets; we are specifically interested in the possibility of extending the logic programming paradigm to the fuzzy case. Several heuristic approaches have been suggested to extend logic programming to the fuzzy case; however, the lack of a foundational base is an obstacle for a wider acceptance of these models, and further, formal approaches typically build upon conventional terms. For instance, restricting to finitely many truth values, a many-valued predicate calculus was proposed in [2].

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This paper is motivated by the introduction of $L$-fuzzy set categories given in [1] and in our interest in extending the use of categorical methods in logic programming to the many-valued case—our long term goal being the generalisation of the categorical unification algorithm given for classical logic by Rydeheard and Burstall in [3], where most general unifiers of a set of terms are built as coequalisers in the Kleisli category associated with the term monad. Following this idea, our approach to the fuzzification of a “set of terms” will be considering a “fuzzy set” of terms; therefore, the different generalizations of the powerset functor become important, especially if the composition of these fuzzy powerset functors with the term monad can be extended to a monad.

In [4], it was shown how set functors can be composed to providing monads, and some motivation to investigate techniques for constructing new monads from given ones was presented. In this work, we introduce a number of set functors which extend the crisp powerset functor, together with their extension principle. Then, $L$-fuzzy set categories are defined for each of these extended powerset functors, and the rationality of the extension principle is proved in the categorical sense; i.e., the associated $L$-fuzzy set categories are equivalent to the category of sets and mappings. Finally, in the last section, each of these new set functors are given a structure of monad and, furthermore, are shown to be useful to build new monads when composed with the term monad.

2. PRELIMINARY DEFINITIONS

Recall the standard definition of the (crisp) power set of a set $X$ by means of characteristic functions

$$
P(X) = \{ A | A \subseteq X \} = \{ A | A : X \rightarrow \{0, 1\} \}.
$$

With this definition, each mapping $A : X \rightarrow \{0, 1\}$ defines a subset of $X$ as the inverse image $A^{-1}(1)$.

This definition can be relaxed by allowing each element to have a degree of membership, using the real unit interval as the codomain of the extended characteristic functions; that is, the fuzzy power set is defined as

$$
\mathcal{F}(X) = \{ A | A : X \rightarrow [0, 1] \}.
$$

Goguen [5] further generalizes this construction by allowing the range of these extended characteristic functions to be a completely distributive lattice, and defining the $L$-fuzzy power set as follows

$$
\mathcal{L}(X) = \{ A | A : X \rightarrow L \}.
$$

Extension Principles

Given two sets $X, Y$ and a mapping $f : X \rightarrow Y$, it is possible to define a mapping between the corresponding power sets $\hat{f} : \mathcal{P}X \rightarrow \mathcal{P}Y$ by means of the direct image of $f$; that is, given $A \in \mathcal{P}(X)$, then $\hat{f}(A) = f(A) \in \mathcal{P}(Y)$.

The extension of $f$ given above admits different generalizations when we are working in the fuzzy case according to an optimistic or pessimistic interpretation of the fuzziness degree.

1. Maximal extension principle: $\hat{f}_M : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is defined such that given $A \in \mathcal{F}(X)$ and $y \in Y$, then $\hat{f}_M(A)(y) = \sup\{ A(x) | x \in f^{-1}(y) \text{ and } A(x) > 0 \}$ if the set is nonempty and $\hat{f}_M(A)(y) = 0$ otherwise.

2. Minimal extension principle: $\hat{f}_m : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is defined such that given $A \in \mathcal{F}(X)$ and $y \in Y$, then $\hat{f}_m(A)(y) = \inf\{ A(x) | x \in f^{-1}(y) \text{ and } A(x) > 0 \}$ if the set is nonempty and $\hat{f}_m(A)(y) = 0$ otherwise.

\footnote{The condition $A(x) > 0$ can be neglected in the maximal extension principles, but not in the minimal ones. For uniformity, we have used a similar representation for both.}
It is straightforward to show that both extensions \( \hat{f}_M \) and \( \hat{f}_m \) coincide with the direct image extension in the case of crisp subsets; that is, given \( A \in \mathcal{P}(X) \subseteq \mathcal{F}(X) \), then \( \hat{f}_M(A) = \hat{f}_m(A) = f(A) \in \mathcal{P}(Y) \subseteq \mathcal{F}(Y) \).

The maximal and minimal extension principles just introduced can be further generalized to the \( L \)-fuzzy power sets, just changing the calculations of suprema and infima by the lattice join and meet operators. In the following, we will use the following set

\[ I = \{ x \in X \mid x \in f^{-1}(y) \text{ and } A(x) > 0 \} \]

(1) maximal \( L \)-fuzzy extension principle: \( \hat{f}_M : \mathcal{L}(X) \to \mathcal{L}(Y) \) is defined in such a way that given \( A \in \mathcal{L}(X) \), then \( \hat{f}_M(A)(y) = \bigvee_i A(x) \) if the set \( I \) is nonempty and \( \hat{f}_M(A)(y) = 0 \) otherwise;

(2) minimal \( L \)-fuzzy extension principle: \( \hat{f}_m : \mathcal{L}(X) \to \mathcal{L}(Y) \) is defined in such a way that given \( A \in \mathcal{L}(X) \), then \( \hat{f}_m(A)(y) = \bigwedge_i A(x) \) if the set \( I \) is nonempty and \( \hat{f}_m(A)(y) = 0 \) otherwise.

3. CATEGORIES OF \( L \)-FUZZY SETS

The extension principles just stated suggest the possibility of extending the definition of \( \mathcal{L} \) to be a functor between \( \text{Set} \), the category of crisp sets and mappings, and \( L \)-fuzzy sets but, obviously, the first step should be to define the appropriate concept of \( L \)-fuzzy set category. The natural way to build a categorical structure on the classes of \( L \)-fuzzy sets is to consider the arrows between \( \mathcal{L}(X) \) and \( \mathcal{L}(Y) \) as those given by any of the extension principles introduced above. Choosing the mappings given by the maximal extension principle, we get to the following definition.

**DEFINITION 1. CATEGORY OF \( L \)-FUZZY SETS.** Let \( L \) be a completely distributive lattice. The category of \( L \)-fuzzy sets has as objects the class \( \{ \mathcal{L}(X) \mid X \in \text{Set} \} \). The set of morphisms between two objects \( \mathcal{L}(X) \) and \( \mathcal{L}(Y) \) is defined as

\[ \{ \hat{f}_M : \mathcal{L}(X) \to \mathcal{L}(Y) \mid f : X \to Y \text{ is a mapping in Set} \} . \]

It is straightforward to check that the previous construction is indeed a category, denoted \( \mathcal{L}\text{-Set} \), which has been previously used in \([4,6]\).

Now that we have a category of fuzzy sets, we can attempt the definition of a functor between \( \text{Set} \) and \( \mathcal{L}\text{-Set} \).

**DEFINITION 2.** Let \( L \) be a completely distributive lattice. The covariant \( L \)-fuzzy power-set functor \( \mathcal{L} \) is obtained by defining \( \mathcal{L}(X) = L^X \), i.e., the \( L \)-fuzzy sets \( A : X \to L \), and following \([5]\), for a mapping \( f : X \to Y \) in \( \text{Set} \), by defining the morphism \( \mathcal{L}f : \mathcal{L}(X) \to \mathcal{L}(Y) \) as

\[ \mathcal{L}f(A)(y) = \bigvee_{f(x) = y} A(x) . \]

The functoriality of the previous construction can be seen in \([5]\). Also, note that the above definition of \( \mathcal{L}f \) is the same than that given by the extension principle, \( \hat{f}_M \).

When working with the minimal extension principle, that is, changing \( \hat{f}_M \) into \( \hat{f}_m \), some problems arise when checking the axioms of category. This is mainly due to the fact that the minimal extension principle is not exactly the dual of the maximal one: in the case that the set \( I \) in the definition is empty, then both \( \hat{f}_M(A)(y) \) and \( \hat{f}_m(A)(y) \) yield the minimum element in the lattice; this is natural in the maximal extension principle, for \( \bigvee \emptyset = 0 \), but \( \bigwedge \emptyset \neq 0 \). This is why it is interesting to consider, in analogy with the results in \([1]\), the \( \alpha \)-upper \( L \)-fuzzy set categories and the \( \alpha \)-lower \( L \)-fuzzy set categories.
DEFINITION 3. For all $\alpha \in L$, the classes of the $\alpha$-upper $L$-fuzzy sets and the $\alpha$-lower $L$-fuzzy sets, denoted $L_\alpha(X)$ and $L^\alpha(X)$ respectively, are defined as follows

\[
\begin{align*}
L_\alpha(X) &= \{ A \mid A \in L(X), A(x) \geq \alpha \text{ or } A(x) = 0, \text{ for all } x \in X \}, \\
L^\alpha(X) &= \{ A \mid A \in L(X), A(x) \leq \alpha \text{ or } A(x) = 1, \text{ for all } x \in X \}.
\end{align*}
\]

DEFINITION 4. CATEGORY OF $L_\alpha$-FUZZY SETS. Let $L$ be a completely distributive lattice and $\alpha \in L$ with $\alpha > 0$. The category of $L_\alpha$-fuzzy sets, denoted $\text{L}_\alpha\text{-Set}$, has as objects the class $\{L_\alpha(X) \mid X \in \text{Set}\}$. The set of morphisms between two objects $L_\alpha(X)$ and $L_\alpha(Y)$ is defined as

\[
\{ f^*: L_\alpha(X) \rightarrow L_\alpha(Y) \mid f : X \rightarrow Y \text{ is a mapping in } \text{Set} \},
\]

where $f^*$ is the restriction of the mapping given by the minimal extension principle to the $L$-fuzzy set $L_\alpha(X)$.

In the theorem below, we check that this construction verifies the axioms of a category.

THEOREM 1. $\text{L}_\alpha\text{-Set}$ is a category.

PROOF. To begin with, let us show that $\tilde{g}_m \circ \tilde{f}_m = (g \circ f)_m$, where $\tilde{f}_m : L_\alpha(X) \rightarrow L_\alpha(Y)$ and $\tilde{g}_m : L_\alpha(Y) \rightarrow L_\alpha(Z)$ are given by applying the minimal extension principle to mappings $f : X \rightarrow Y$ and $g : Y \rightarrow Z$.

We will proceed by cases (1),(2).

(1) Assuming that $\tilde{g}_m(\tilde{f}_m(A))(z) = 0$, then we have two cases.

(a) $g^{-1}(z) = \emptyset$.

In this case $(g \circ f)^{-1}(z) = \emptyset$ and then $(g \circ f)_m(A)(z) = 0$.

(b) $\tilde{f}_m(A)(y) = 0$ for all $y \in g^{-1}(z)$.

Here, for all $y \in g^{-1}(z)$ we have either $f^{-1}(y) = \emptyset$ or $A(x) = 0$ for all $x \in f^{-1}(y)$.

Assuming that $f^{-1}(y) = \emptyset$ for all $y \in g^{-1}(z)$, then $(g \circ f)^{-1}(z) = \emptyset$ and, therefore, $(g \circ f)_m(A)(z) = 0$. Otherwise, we would have $(g \circ f)^{-1}(z) \neq \emptyset$, but also $A(x) = 0$ for all $x \in (g \circ f)^{-1}(z)$ and we would also have $(g \circ f)_m(A)(z) = 0$ in this case.

Therefore, if $\tilde{g}_m(\tilde{f}_m(A))(z) = 0$, then $(g \circ f)_m(A)(z) = 0$.

(2) Now, assume $\tilde{g}_m(\tilde{f}_m(A))(z) > 0$ and consider the following equalities:

\[
\tilde{g}_m(\tilde{f}_m(A))(z) = \bigwedge \{ \tilde{f}_m(A)(y) \mid y \in g^{-1}(z), \tilde{f}_m(A)(y) > 0 \} = \bigwedge \{ A(x) \mid x \in f^{-1}(y), A(x) > 0, y \in g^{-1}(z), \tilde{f}_m(A)(y) > 0 \}
\]

\[
= \bigwedge \left\{ A(x) \mid x \in \bigcup_{y \in g^{-1}(z)} f^{-1}(y), A(x) > 0 \right\} = \bigwedge \left\{ A(x) \mid x \in \bigcup_{y \in g^{-1}(z)} f^{-1}(y), A(x) > 0 \right\} = (g \circ f)_m(A)(z).
\]

The equality $(\ast)$ holds because of the hypothesis $\alpha > 0$, for in this case if $\tilde{f}_m(A)(y) = 0$, then $f^{-1}(y) = \emptyset$. Thus, we have that if $\tilde{g}_m(\tilde{f}_m(A))(z) > 0$, then $\tilde{g}_m(\tilde{f}_m(A))(z) = (g \circ f)_m(A)(z)$.
The two cases altogether show that \( g_m \circ \hat{f}_m = (\tilde{g} \circ \tilde{f})_m \), and this fact allows us easily to obtain the axioms of category.

**Remark 1.** It is important to note that if \( \alpha = 0 \), we do not get a category. This is due to the fact that if \( \hat{f}_m(A)(y) = 0 \), then \( f^{-1}(y) \) is not necessarily empty. A counterexample can be found in [1].

Now, the definition of the \( L_\alpha \) functor between \( \text{Set} \) and \( L_\alpha\text{-Set} \) is straightforward.

**Definition 5.** Let \( L \) be a completely distributive lattice and \( \alpha \in L, \alpha > 0 \). The covariant \( \alpha \)-upper \( L \)-fuzzy power-set functor, \( L_\alpha : \text{Set} \to L_\alpha\text{-Set} \), is obtained by defining \( L_\alpha(X) \) as in Definition 3 and by defining \( L_\alpha f = \hat{f}_m \) for each morphism \( f : X \to Y \) in \( \text{Set} \).

Now, it is easy to check that the above definition is really a functor, for we have already proved that \( L_\alpha(f \circ g) = L_\alpha(f) \circ L_\alpha(g) \) in the proof of Theorem 1 above. It follows immediately from the definition that \( L_\alpha(1_X) = 1_{L_\alpha X} \).

The process above can be almost literally dualised, when considering the \( \alpha \)-lower sets and the maximal extension principle.

**Definition 6. Category of \( L^\alpha \)-Fuzzy Sets.** Let \( L \) be a completely distributive lattice and \( \alpha \in L. \) The category of \( L^\alpha \)-fuzzy sets, denoted \( L^\alpha\text{-Set} \), has as objects the class \( \{ L^\alpha(X) \mid X \in \text{Set} \} \). The set of morphisms between two objects \( L^\alpha(X) \) and \( L^\alpha(Y) \) is defined as

\[
\left\{ \hat{f}_M : L^\alpha(X) \to L^\alpha(Y) \mid f : X \to Y \text{ is a mapping in } \text{Set} \right\}.
\]

**Theorem 2.** \( L^\alpha\text{-Set} \) is a category.

**Proof.** The proof follows the steps of Theorem 1. \( \blacksquare \)

**Remark 2.** Note that in this case, no problem arises when \( \alpha = 1 \). Actually, the category \( L^1\text{-Set} \) is equal to \( L\text{-Set} \).

**Definition 7.** Let \( L \) be a completely distributive lattice and \( \alpha \in L. \) The covariant \( \alpha \)-lower \( L \)-fuzzy power-set functor, \( L^\alpha : \text{Set} \to L^\alpha\text{-Set} \), is defined for objects as in Definition 3, and for a morphism \( f : X \to Y \) in \( \text{Set} \) is defined as \( L^\alpha f = \hat{f}_M \).

It is easy to check that the above definition is really a functor.

The Rationality of Extension Principles

Regarding extension principles for fuzzy sets, it is important to check the rationality of the extension. In a categorical context, this amounts to showing that the extended categories are essentially the category of sets, or in more technical words, that the extended category is categorically equivalent to the category of sets and mappings.

To begin with, we recall the following characterisation of equivalent categories; see [7] for a proof.

**Lemma 1.** Two categories \( C_1 \) and \( C_2 \) are equivalent if and only if there exists a functor \( \Phi : C_1 \to C_2 \) such that

1. for all pairs \( A, B \) in \( C_1 \), we have a bijection between the sets \( \text{Hom}_{C_1}(A, B) \) and \( \text{Hom}_{C_2}(\Phi(A), \Phi(B)) \);
2. for all object \( A' \) in \( C_2 \), there exists an object \( A \) in \( C_1 \), such that \( \Phi(A) \) and \( A' \) are isomorphic objects in \( C_2 \).

**Theorem 3.** The categories \( \text{Set}, L_\alpha\text{-Set}, \) and \( L^\alpha\text{-Set} \) are equivalent.

**Proof.** The definition of the category \( L_\alpha\text{-Set} \) suggests to consider \( \Phi \) as the functor \( L_\alpha \) in Definition 5. In addition, the hypotheses of Lemma 1 follow directly from the definition of \( L_\alpha\text{-Set} \). Therefore, \( \text{Set} \) and \( L_\alpha\text{-Set} \) are equivalent categories.

The equivalence between \( \text{Set} \) and \( L^\alpha\text{-Set} \) is proved in a similar way. \( \blacksquare \)
The rationality of these categories allows the definition of a structure of monad on the corresponding set functors, modulo the equivalence of categories; in other words, the fact that \( \mathcal{L}_\alpha \) and \( \mathcal{L}^\alpha \) are endofunctors raises the question whether they can be extended to monads.

4. \( L \)-FUZZY POWERSET FUNCTORS AS MONADS

A monad can be seen as the abstraction of the concept of adjoint functors and, in a sense, an abstraction of universal algebra. It is interesting to note that monads are useful not only in universal algebra, but it is also an important tool in topology when handling regularity, iteratedness, and compactifications, and also in the study of toposes and related topics.

DEFINITION 8. Let \( C \) be a category. A monad (or triple, or algebraic theory) over \( C \) is written as \( \Phi = (\Phi, \eta, \mu) \), where \( \Phi : C \to C \) is a (covariant) functor, and \( \eta : 1 \to \Phi \) and \( \mu : \Phi \circ \Phi \to \Phi \) are natural transformations for which \( 1 \circ \mu = \mu \circ \Phi \eta = \eta \circ \Phi = \text{id}_\Phi \) hold.

In the particular case of the functor \( L \), Manes proved in [8] that \( (\mathcal{L}, \eta, \mu) \) with unit \( \eta_X : X \to \mathcal{L}X \) defined by

\[
\eta_X(x)(x') = \begin{cases} 1, & \text{if } x = x', \\ 0, & \text{otherwise,} \end{cases}
\]

and \( \mu_X : \mathcal{L}\mathcal{L}X \to \mathcal{L}X \) by

\[
\mu_X(A)(x) = \bigvee_{A \in \mathcal{L}X} A(x) \wedge A(A)
\]

is a monad. This result can be easily shifted to the case of \( (\mathcal{L}^\alpha, \eta^\alpha, \mu^\alpha) \).

Functors \( \mathcal{L}_\alpha \) are more interesting to study and can also be provided with a monad structure \( (\mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha) \) where the unit \( \eta_\alpha X : X \to \mathcal{L}_\alpha X \) is defined as in the case \( \eta^\alpha \), by

\[
\eta_\alpha X(x)(x') = \begin{cases} 1, & \text{if } x = x', \\ 0, & \text{otherwise,} \end{cases}
\]

and the multiplication \( \mu_\alpha X : \mathcal{L}_\alpha \mathcal{L}_\alpha X \to \mathcal{L}_\alpha X \) is defined by

\[
\mu_\alpha X(A)(x) = \begin{cases} \bigwedge_{A \in I} A(x) \wedge A(A), & \text{if } I = \{A \in \mathcal{L}_\alpha X \mid A(x) \wedge A(A) > 0\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]

Let us show that we have two natural transformations.

LEMMA 2. \( \eta_\alpha : 1 \to \mathcal{L}_\alpha \) and \( \mu_\alpha : \mathcal{L}_\alpha \mathcal{L}_\alpha \to \mathcal{L}_\alpha \) are natural transformations.

PROOF. In the following, we will drop the subscript \( \alpha \) in \( L, \mu, \) and \( \eta \).

We have only to prove \( \mu \) is a natural transformation, that is

\[
\mu_Y ((\mathcal{L}\mathcal{L}f)(A)) (y) = \mathcal{L}f(\mu_X(A))(y).
\]

Consider \( J = \{B \in \mathcal{L}Y \mid B(y) > 0, \mathcal{L}\mathcal{L}f(A)(B) > 0\} \); then

\[
\mu_Y ((\mathcal{L}\mathcal{L}f)(A))(y) = \begin{cases} \bigwedge_{B \in J} B(y) \wedge \mathcal{L}\mathcal{L}f(A)(B), & \text{if } J \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]

That is, \( \mu_Y ((\mathcal{L}\mathcal{L}f)(A))(y) = 0 \) iff one of the following conditions hold:

- \( B(y) = 0 \) for all \( B \),
- \( \mathcal{L}\mathcal{L}f(A)(B) = 0 \) for all \( B \) satisfying that \( B(y) > 0 \).

Under these conditions, it is easy to check that \( \mathcal{L}f(\mu_X(A))(y) = 0 \).
Now, assume that there exists \( B \) such that \( B(y) > 0 \) and \( \mathcal{L} \mathcal{L} \mathcal{f}(A)(B) > 0 \); specifically, by the definition of \( \mathcal{L} \), there exists \( A \in \mathcal{L}(X) \) such that \( \mathcal{L} \mathcal{f}(A) = B \) and \( \mathcal{A}(A) > 0 \); in this case, we also have that \( B(y) = \mathcal{L} \mathcal{f}(A)(y) > 0 \) and, by definition of \( \mathcal{L} \), there exists \( x \in X \) satisfying \( \mathcal{A}(x) > 0 \) and \( f(x) = y \); therefore, we have \( \mu_X(A(x)) > 0 \).

\[
\mu_Y((\mathcal{L} \mathcal{L} \mathcal{f})(A))(y) = \bigwedge_{\substack{B \in \mathcal{L} \mathcal{Y} \\ B(y) > 0 \\ \mathcal{L} \mathcal{L} \mathcal{f}(A)(B) > 0}} B(y) \wedge \mathcal{L} \mathcal{L} \mathcal{f}(A)(B)
\]

\[
= \bigwedge_{\substack{B \in \mathcal{L} \mathcal{Y} \\ B(y) > 0 \\ \mathcal{L} \mathcal{L} \mathcal{f}(A)(B) > 0}} B(y) \wedge \left( \bigwedge_{\substack{\mathcal{A} \in (\mathcal{L} \mathcal{f})^{-1}(B) \\ \mathcal{A}(A) > 0}} \mathcal{A}(A) \right)
\]

\[
= \bigwedge_{\substack{B \in \mathcal{L} \mathcal{Y} \\ B(y) > 0 \\ \mathcal{A} \in (\mathcal{L} \mathcal{f})^{-1}(B) \\ \mathcal{A}(A) > 0}} B(y) \wedge \mathcal{A}(A)
\]

\[
= \bigwedge_{\substack{\mathcal{A} \in \mathcal{L}(X) \\ \mathcal{L} \mathcal{f}(A)(y) > 0 \\ \mathcal{A}(A) > 0}} \mathcal{L} \mathcal{f}(A)(y) \wedge \mathcal{A}(A)
\]

\[
= \bigwedge_{\substack{\mathcal{A} \in \mathcal{L}(X) \\ \mathcal{L} \mathcal{f}(A)(y) > 0 \\ \mathcal{A}(A) > 0}} \left( \bigwedge_{\mathcal{z} \in f^{-1}(y) \\ \mathcal{A}(x) > 0} \mathcal{A}(x) \right) \wedge \mathcal{A}(A)
\]

\[
= \bigwedge_{\substack{\mathcal{A} \in \mathcal{L}(X) \\ \mathcal{L} \mathcal{f}(A)(y) > 0 \\ \mathcal{A}(A) > 0}} \mathcal{A}(x) \wedge \mathcal{A}(A)
\]

\[
= \bigwedge_{\substack{\mathcal{A} \in \mathcal{L}(X) \\ \mathcal{L} \mathcal{f}(A)(y) > 0 \\ \mathcal{A}(A) > 0}} (A(x) \wedge \mathcal{A}(A)) = \mathcal{L} \mathcal{f}(\mu_X(A))(y).
\]

As stated previously, the difference between \( \mathcal{L}^\alpha \) and \( \mathcal{L}_\alpha \) is not only a matter of duality; therefore, it is worthwhile to give a detailed proof that \( \mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha \) is a monad.

**Theorem 4.** \( \mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha \) is a monad.

**Proof.** The left unit identity requires us to prove that \( \mu_X(\eta \mathcal{L} \mathcal{X}(A))(x) = A(x) \), for all \( x \in X \) and \( A \in \mathcal{L}X \).

Consider \( I = \{ B \in \mathcal{L}X \mid B(x) \wedge \eta \mathcal{L} \mathcal{X}(A)(B) > 0 \} \); then

\[
\mu_X(\eta \mathcal{L} \mathcal{X}(A))(x) = \begin{cases} 
\bigwedge_{B \in I} B(x) \wedge \eta \mathcal{L} \mathcal{X}(A)(B), & \text{if } I \neq \emptyset, \\
0, & \text{otherwise},
\end{cases}
\]

\[
= \begin{cases} 
A(x), & \text{if } A(x) > 0, \\
0, & \text{if } A(x) = 0,
\end{cases}
\]

so \( \mu_X(\eta \mathcal{L} \mathcal{X}(A))(x) = A(x) \).

The equality \( (*) \) follows from the definition of \( \eta \), for \( \eta \mathcal{L} \mathcal{X}(A)(B) > 0 \) if and only if \( A = B \).

For the right unit identity, we have to prove that \( \mu_X(\mathcal{L} \eta \mathcal{X}(A))(x) = A(x) \), for all \( x \in X \) and \( A \in \mathcal{L}X \). Given \( I = \{ B \in \mathcal{L}X \mid B(x) \wedge \eta \mathcal{L} \mathcal{X}(A)(B) > 0 \} \), we have

\[
\mu_X(\mathcal{L} \eta \mathcal{X}(A))(x) = \begin{cases} 
\bigwedge_{B \in I} B(x) \wedge \eta \mathcal{L} \mathcal{X}(A)(B), & \text{if } I \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}
\]

The equality \( (*) \) follows from the definition of \( \eta \), for \( \eta \mathcal{L} \mathcal{X}(A)(B) > 0 \) if and only if \( A = B \).
Thus, \( \mu_X(\mathcal{L}\eta_X(A))(x) > 0 \) iff there exist \( B \in \mathcal{L}X \) and \( y \in \eta_X^{-1}(B) \) with \( B(x) \) and \( A(y) > 0 \); in this case,

\[
\mu_X(\mathcal{L}\eta_X(A))(x) = \bigwedge_{\substack{B \in \mathcal{L}X \ni B(x) > 0 \in \eta_X^{-1}(B) \ni \eta_X(A)(B) > 0 \in A(y) > 0}} B(x) \land \mathcal{L}\eta_X(A)(B) - \bigwedge_{\substack{B \in \mathcal{L}X \ni B(x) > 0 \in \eta_X^{-1}(B) \ni \eta_X(A)(B) > 0 \in A(y) > 0}} D(x) \land \bigwedge_{\substack{y \in \eta_X^{-1}(B) \ni A(y) > 0}} A(y) = \bigwedge_{\substack{B \in \mathcal{L}X \ni B(x) > 0 \in \eta_X^{-1}(B) \ni A(y) > 0}} B(x) \land A(y) = A(x).
\]

For the last equality, notice that for each \( B \) with \( B(x) > 0 \) and \( y \in \eta_X^{-1}(B) \), we have that \( B(x) = \eta_X(y)(x) > 0 \); thus, necessarily \( y = x \) and \( B(x) = 1 \).

For the associativity of \( \mu \), we have to prove that

\[
\mu_X(\mathcal{L}\mu_X(A))(x) = \mu_X(\mu_X(A))(x),
\]

for all \( x \in X \) and \( A \in \mathcal{L}LX \).

Consider the set \( I = \{ A \in \mathcal{L}(X) \mid A(x) \land \mathcal{L}\mu_X(A), \text{ for all } x \in X \text{ and } \mathcal{L} \}

\[
\mu_X(\mathcal{L}\mu_X(A))(x) = \left\{ \begin{array}{ll}
\bigwedge_{A \in I} A(x) \land \mathcal{L}\mu_X(A), & \text{if } I \neq \emptyset, \\
0, & \text{otherwise.}
\end{array} \right.
\]

Therefore, \( \mu_X(\mathcal{L}\mu_X(A))(x) = 0 \) if and only if the following condition holds.

For all \( A \in \mathcal{L}X \) such that \( A(x) > 0 \) and all \( A \in \mu_X^{-1}(A) \), we have that \( A(x) > 0 \).

Now, if \( \mu_X(\mathcal{L}\mu_X(A))(x) > 0 \), then there exists \( A \in \mathcal{L}X \) such that

- \( A(x) > 0 \);
- there exists \( A \in \mu_X^{-1}(A) \) such that \( A(x) > 0 \).

Specifically, since \( \mu_X(A) = A \), we have \( \mu_X(A)(x) = A(x) > 0 \), and therefore, by definition of \( \mu_X \), there exists \( B \) such that \( B(x) > 0 \) and \( A(B) > 0 \) and \( \mu_X(A)(B) > 0 \). As a result, the subscripts used in the following calculations of meets are all nonempty (which allows us to use just the associativity of meets):

\[
\mu_X(\mathcal{L}\mu_X(A))(x) = \bigwedge_{\substack{A \in \mathcal{L}X \ni A(x) > 0 \in \mu_X^{-1}(A) \ni A(B) > 0 \in A(y) > 0}} A(x) \land \bigwedge_{\substack{A \in \mu_X^{-1}(A) \ni A(y) > 0 \in A(z) > 0}} A(z) = \bigwedge_{\substack{A \in \mathcal{L}X \ni A(x) > 0 \in \mu_X^{-1}(A) \ni A(B) > 0 \in A(y) > 0}} A(x) \land \bigwedge_{\substack{A \in \mu_X^{-1}(A) \ni A(y) > 0 \in A(z) > 0}} A(z) = \bigwedge_{\substack{A \in \mathcal{L}X \ni A(x) > 0 \in \mu_X^{-1}(A) \ni A(B) > 0 \in A(y) > 0}} \left( \bigwedge_{\substack{B \in \mathcal{L}X \ni B(x) > 0 \in \mu_X^{-1}(A) \ni A(B) > 0 \in A(y) > 0}} B(x) \land A(B) \right) \land A(A).
\]
Therefore, we have proved that, if $\mu_X(Ł \mu_X(Å))(x) > 0$, then

$$
\mu_X(Ł \mu_X(Å))(x) = \mu_X(Ł \mu_X(Å))(x).
$$

To finish the proof, it suffices to prove that

if $\mu_X(Ł \mu_X(Å))(x) > 0$, then $\mu_X(Ł \mu_X(Å))(x) > 0$.

Consider $J = \{B \in ŁX \mid B(Å) > 0, Ł(Å)(A) > 0\}$

$$
\mu_X(Ł(Å))(x) = \begin{cases}
\bigwedge_{B \in J} B(x) \wedge Ł(Å)(B), & \text{if } J \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}
$$

Thus, if $\mu_X(Ł(Å))(x) > 0$, then there exists $B \in ŁX$ and $A \in ŁŁX$, such that $B(x) > 0$, $Ł(Å)(A) > 0$, and $Ł(Å)(A) > 0$. Now, if we consider $A = \mu_X(Å)$, then the following conditions are satisfied:

- $0 < A(x) = \mu_X(Å)(x)$, because $B(x) > 0$, and $Ł(Å)(A) > 0$.
- $A \in \mu_X^{-1}(Å)$ and $Ł(Å)(A) > 0$.

Therefore, we can conclude that $\mu_X(Ł \mu_X(Å))(x) > 0$ as well.

5. COMPOSING L-FUZZY POWERSET MONADS AND THE TERM MONAD

As stated in the introduction, the use of categorical tools in logic programming leads naturally to the problem of categorically expressing fuzzy sets of generalized terms, in order to achieve a categorical paradigm of fuzzy logic programming.

There are different possibilities to get a fuzzy set of generalized terms. Extending the idea by Rydeheard and Burstall in [3] which considered most general unifiers of a set of terms as coequalisers in the Kleisli category associated with the term monad, one naturally has to consider the composition of the $Ł$-fuzzy powerset monads and the term monad. Specifically, we have to check whether the composition of these fuzzy powerset monads with the term monad can be extended to a monad.

To begin with, we will briefly introduce the functorial presentation of the set of terms on a signature.

5.1. The Term Monad

Regarding the set of terms, it is useful to adopt a more functorial presentation of it, as opposed to using the conventional inductive definition of terms, where we bind ourselves to certain styles of proofs. Even if a purely functorial presentation might seem complicated, there are advantages when we define corresponding monads, and further, a functorial presentation simplifies efforts to prove results concerning compositions of monads.
In this work, we adopt an intermediate approach, for our main emphasis is not focused on the
categorical extension of the term algebra, but on the results concerning composition of monads.
The completely categorical approach can be found in [9], which was used also in [4].

Let \( \Omega = \bigcup_{n=0}^{\infty} \Omega_n \) be an operator domain, where each \( \Omega_n \) is intended to contain operators of
arity \( n \). The definition of the term functor \( T_\Omega : \text{Set} \to \text{Set} \) is given as \( T_\Omega(X) = \bigcup_{k=0}^{\infty} T_\Omega^k(X) \),
where

1. \( T_\Omega^0(X) = X; \)
2. \( T_\Omega^{k+1}(X) = \{(n, \omega, (x_i)_{i \leq n}) \mid \omega \in \Omega_n, n \in \mathbb{N}, x_i \in T_\Omega^k(X)\}. \)

The notation \((n, \omega, (x_i)_{i \leq n})\), standing for the term \( \omega(x_1, \ldots, x_n) \), is due to the fact that the
operator domain is defined as a coproduct indexed by \( n \).

Note that \((T_\Omega X, (\sigma_\omega)_{\omega \in \Omega})\) is an \( \Omega \)-algebra, simply defining \( \sigma_\omega((m_i)_{i \leq n}) \) to be the tuple \((n, \omega, (m_i)_{i \leq n})\) for \( \omega \in \Omega_n \) and \( m_i \in T_\Omega X \). Actually, this algebra is a freely generated algebra in the
category of \( \Omega \)-algebras; that is, for an \( \Omega \)-algebra \( B = (Y, (\tau_\omega)_{\omega \in \Omega}) \), a morphism \( f : X \to Y \) in
\( \text{Set} \) can be extended to an \( \Omega \)-homomorphism \( f^* : (T_\Omega X, (\sigma_\omega)_{\omega \in \Omega}) \to (Y, (\tau_\omega)_{\omega \in \Omega}) \), called the
\( \Omega \)-extension of \( f \) associated to \( B \), given by

\[
    f^*[T_\Omega X] = f \quad \text{and} \quad f^*(n, \omega, (m_i)_{i \leq n}) = \tau_\omega \left( (f^*(m_i))_{i \leq n} \right),
\]

for all \( n \in \mathbb{N} \) and \((n, \omega, (m_i)_{i \leq n}) \in T_\Omega^k X \).

A morphism \( X \xrightarrow{f} Y \) in \( \text{Set} \) can also be extended to the corresponding \( \Omega \)-homomorphism
\( (T_\Omega X, (\sigma_\omega)_{\omega \in \Omega}) \xrightarrow{T_\Omega f} (T_\Omega Y, (\tau_\omega)_{\omega \in \Omega}) \), where \( T_\Omega f \) is defined to be the \( \Omega \)-extension of \( f : X \to Y \) associated to \( (T_\Omega Y, (\tau_\omega)_{\omega \in \Omega}) \).

Now, \( T_\Omega = (T_\Omega, \eta^T_\Omega, \mu^T_\Omega) \) is a monad, as shown in [8], with the following definition for the
unit \( \eta^T_\Omega(x) = x \) and, regarding the multiplication, \( \mu^T_\Omega = \text{id}_{T_\Omega X} \) is the \( \Omega \)-extension of the \( \text{id}_{T_\Omega X} \) with
respect to \((T_\Omega X, (\sigma_\omega)_{\omega \in \Omega})\).

5.2. The Composition of an \( L \)-Fuzzy Powerset Monad
and the Term Monad is a Monad

According to the results in [4,10], it is useful to have a swapper transformation in order to
make the composition \( L \circ T \) a monad.

Let us define recursively a mapping \( \sigma_X : T \circ L \circ X \to L \circ T \circ X \).

1. For the base case \( T^0(L_X) = L_X \), if \( \ell \in L_X \), then \( \sigma_X(\ell) = \ell \).
2. For the inductive step, consider \( \ell = (n, \omega, (\ell_i)_{i \leq n}) \); then \( \sigma_X(\ell) : T \circ X \to L \) is defined as

\[
    \sigma_X(\ell)(n', \omega', (m_i)_{i \leq n'}) = \begin{cases} 
        \bigwedge_{i \leq n} \sigma_X(\ell_i)(m_i), & \text{if } n = n' \text{ and } \omega = \omega', \\
        0, & \text{otherwise},
    \end{cases}
\]

that is, \((n', \omega', (m_i)_{i \leq n'})\) is in \( \sigma_X(\ell) \) if \( \omega' = \omega \) and each \( m_i \) is in \( \sigma(\ell_i) \).

As a consequence of the definition, it is easy to check that if \( L = 2 \), then \( \sigma_X(n, \omega, (\ell_i)_{i \leq n}) = \{(n, \omega, (m_i)_{i \leq n}) \mid m_i \in \sigma(\ell_i)\} \).

In the following, we will drop the subscript \( \alpha \), and write \( L \) instead of \( L \circ \alpha \). We have to show that \( \sigma \) is a natural transformation.

**Lemma 3.** \( \sigma : T \circ L \to L \circ T \) is a natural transformation.

**Proof.** Given \( l \in T \circ L \), and \( X \xrightarrow{f} Y \) in \( \text{Set} \) we will show, by structural induction, that \( \sigma_X \circ \sigma_Y(l) = \sigma_Y \circ T \circ L \circ f(l) \).

If \( l \in T^0 \circ L \), then it is immediate.

If \( k > 0 \), then we may write \( l = (n, \omega, (\ell_i)_{i \leq n}) \), where \( \ell_i \in T^{k_i} \circ L \), \( k_i < k \), for all \( i \leq n \). Let us check that \( \sigma_X \circ \sigma_Y(l)(s) = \sigma_Y \circ T \circ L \circ f(l)(s) \), for all \( s \in T \circ Y \).
For the base case in which $s \in Y$, it is straightforward, and in the general case we can assume $s = (n, \omega, (s_i)_{i \leq n}) \in TY$. Recall that

$$
\mathcal{L}Tf(\sigma_X(l))(s) = \begin{cases} 
\bigwedge_{t \in I} \sigma_X(l)(t), & \text{if } I \neq \emptyset, \\
0, & \text{otherwise}, 
\end{cases}
$$

$$
= \begin{cases} 
\bigwedge_{t \in I \leq n} \sigma_X(l_i)(t_i), & \text{if } I_i \neq \emptyset \text{ for all } i \leq n, \\
0, & \text{otherwise}, 
\end{cases}
$$

where $I = \{t = (n, \omega, (t_i)_{i \leq n}) \in TX \mid (Tf)(t) = s \text{ and } \sigma_X(l)(t) > 0\}$ and $I_i = \{t_i \in TX \mid (Tf)(t_i) = s_i \text{ and } \sigma_X(l_i)(s_i) > 0\}$.

Now, we have the following chain of equalities:

$$
\sigma_Y(\mathcal{L}f(l))(s) = \bigwedge_{i \leq n} \sigma_Y(\mathcal{L}f(l_i))(s_i) = \bigwedge_{i \leq n} \mathcal{L}Tf(\sigma_X(l_i))(s_i)
$$

$$
= \begin{cases} 
\bigwedge_{t \in I \leq n} \sigma_X(l_i)(t_i), & \text{if } I_i \neq \emptyset \text{ for all } i \leq n, \\
0, & \text{otherwise}, 
\end{cases}
$$

where the equality (*) follows by the induction hypothesis.

The just defined swapper transformation allows us to define natural nominates for the unit and multiplication of the composed monad

$$
\eta_{X^T}^\sigma = \eta_{TX}^\sigma \circ \eta_X^\sigma,
$$

$$
\mu_{X^T}^\sigma = \mathcal{L} \mu_T \circ \mu_{TX}^\sigma \circ \sigma \mathcal{L} X = \mu_{TX} \circ \mathcal{L} \mu_{TX} \circ \sigma \mathcal{L} X.
$$

By using Theorem 2 in [10], we have just to check the following three properties of $\sigma$ w.r.t. the unit and the multiplication defined above.

1. $\sigma \circ \eta_{X^T}^\sigma = \mathcal{L} \eta^T \sigma$

2. $\sigma \circ \eta^T \sigma \mathcal{L} = \mathcal{L} \eta^T$

3. $\sigma \circ \tau \eta^T \sigma = \eta_{X^T}$

**Lemma 4.** The swapper $\sigma$ satisfies equation (2a); that is, $\sigma \circ \eta_{X^T} = \eta_{X^T}$.

**Proof.** Just recall the definitions: $\eta$ in the base case is $\eta_X^T = T^0 X = X$, and $\eta_{X^T}^X = T^0 \mathcal{L} X = \mathcal{L} X$ in the recursive step, and $\sigma_X$ on $\mathcal{L} X$ is $\mathcal{L} X$.

**Lemma 5.** The swapper $\sigma$ satisfies equation (3a); that is, $\sigma \circ \tau \eta^T \sigma = \eta_{X^T}$.

**Proof.** We have only to prove that, for $m \in TX$, we have $\sigma_X(\mathcal{L} \eta_X^T(m)) = \eta_{X^T}(m)$.

1. If $m \in X$, then $\mathcal{L} \eta_X^T(m) = \eta_X^T(m) \in T^0 \mathcal{L} X$; therefore, $\sigma_X(\mathcal{L} \eta_X^T(m)) = \eta_X^T(m) = \eta_{X^T}(m)$.

2. If $m = (n, \omega, (m_i)_{i \leq n})$, then $\sigma_X(\mathcal{L} \eta_X^T(m)) : TX \rightarrow L_i$; consider $m' \in TX$. The only case in which $\sigma_X(\mathcal{L} \eta_X^T(m))(m')$ can be nonzero arises when $m' = (n, \omega, (m'_i)_{i \leq n})$.

$$
\sigma_X(\mathcal{L} \eta_X^T(m))(m') \overset{\text{Def}}{=} \mathcal{L} T f(\sigma_X(n, \omega, (\mathcal{L} \eta_X^T(m_i))_{i \leq n}))(m')
$$

$$
= \bigwedge_{i \leq n} \sigma_X(T \eta_X^T(m_i))(m'_i)
$$

$$
\overset{\text{Induc.}}{=} \bigwedge_{i \leq n} \eta_{X^T}(m_i)(m'_i) = \eta_{X^T}(m)(m').
$$

For the proof of Property (1a), we will use the following technical lemma.
LEMMA 6. Consider the mappings $\mu^T_X$ and $\sigma_X : \mathcal{TLX}$, elements $R \in \mathcal{TLX}$, $m \in TX$, and the set $I = \{ r \in TLX \mid R(r) \land (\sigma_X(r))(m) > 0 \}$; then

$$(\mu^T_X(L\sigma_X(R)))(m) = \begin{cases} \bigwedge_{r \in I} R(r) \land (\sigma_X(r))(m), & \text{if } I \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Assume that there exists $A \in \mathcal{LTX}$ such that $(L\sigma_X(R))(A)$ and $A(m)$ are nonzero (and thus the left-hand side of the equation is also nonzero).

$$(\mu^T_X(L\sigma_X(R)))(m) = \bigwedge_{A \in \mathcal{LTX}, A(m) > 0, R(A) > 0} R(r) \land (\sigma_X(r))(m) \land A(m)$$

for the last equality just note that the elements in $A \in \mathcal{LTX}$ for which there is $r$ satisfying $\sigma_X(r) = A$ are exactly those in the image of $U_X$.

Conversely, let us prove that if there exists an element $r \in \mathcal{TLX}$ such that $R(r) > 0$ and $\sigma_X(r)(m) > 0$, then $(\mu^T_X(L\sigma_X(R)))(m) > 0$. This is not difficult, since we have only to find an $A \in \mathcal{LTX}$ with $A(m) > 0$ and $L\sigma_X(R)(A) > 0$, simply consider $A = \sigma_X(r)$.

LEMMA 7. The swapper $\sigma$ satisfies equation (1o) $L\mu^T \circ \sigma_T \circ T\mu^T \circ TL\sigma = \mu^T \circ L\sigma \circ L\mu^T \circ \sigma_T L$.

PROOF. According to the definition of the multiplication $\mu^T$, we have only to prove that for $d \in \mathcal{TLTLX}$ and $m \in TX$,

$$\sigma_{TX}((T(\mu^T_X \circ L\sigma))(d))(m) = (\mu^T_X \circ L\sigma_X)(\sigma_{TLX}(d))(m).$$

Assume that $\sigma_{TX}((T(\mu^T_X \circ L\sigma))(d))(m) > 0$. By induction, if $d \in \mathcal{TLTLX}$ and $m \in TX$, the equation trivially holds; now consider $d = (n, \omega, (d_i)_{i \leq n})$ and $m = (n, \omega, (m_i)_{i \leq n})$ (otherwise the left-hand side of the equation is zero),

$$\sigma_{TX}((T(\mu^T_X \circ L\sigma))(d))(m) = \bigwedge_{i \leq n} \sigma_{TX}((T(\mu^T_X \circ L\sigma))(d_i))(m_i)$$

Induc. $\bigwedge_{i \leq n} (\mu^T_X \circ L\sigma_X)(\sigma_{TLX}(d_i))(m_i)$

Lem. 6 $\bigwedge_{i \leq n} \bigwedge_{r_i \in \mathcal{TLX}} (\sigma_{TLX}(d_i)(r_i) \land \sigma_X(r_i)(m_i))$
\[ \begin{align*}
\Rightarrow & \quad \bigwedge_{r=(n,\omega,\nu,\gamma) \in \mathcal{T} \cup \mathcal{L}} \left( \sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d)(r) \land \sigma_{\mathcal{X}}(r)(m) \right) \\
\Rightarrow & \quad \bigwedge_{r \in \mathcal{T} \cup \mathcal{L}} \left( \sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d)(r) \land \sigma_{\mathcal{X}}(r)(m) \right) \\
\Rightarrow & \quad \left( \mu_{\mathcal{T} \mathcal{L} \mathcal{X}} \circ \mathcal{L}_{\mathcal{X}} \right) \left( \sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d) \right)(m),
\end{align*} \]

where in \((\ast)\) associativity, commutativity, and idempotence of the infimum is used.

Conversely, if \(\left( \mu_{\mathcal{T} \mathcal{L} \mathcal{X}} \circ \mathcal{L}_{\mathcal{X}} \right)(\sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d))(m) > 0\), then it is easy to check that \(\sigma_{\mathcal{T} \mathcal{X}}((T(\mu_{\mathcal{T} \mathcal{L} \mathcal{X}} \circ \mathcal{L}_{\mathcal{X}}))(d))(m) > 0\); for given \(r \in \mathcal{T} \cup \mathcal{L} \mathcal{X}\) satisfying \(\sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d)(r) > 0\) and \(\sigma_{\mathcal{X}}(r)(m) > 0\), there exists \(A \in \mathcal{L} \mathcal{X}\) (namely, \(\sigma_{\mathcal{X}}(r)\)) with \(A(m) > 0\) and \(\mathcal{L}_{\mathcal{X}}(\sigma_{\mathcal{T} \mathcal{L} \mathcal{X}}(d))(A) > 0\).

6. CONCLUSIONS

We have generalized previous constructions of fuzzy set categories by considering \(L\)-fuzzy sets in which the values of the characteristic functions run on a completely distributive lattice. In addition, \(L\)-fuzzy categories have been defined, using \(L\)-fuzzy sets, and proved to be rational. The final part of the paper has been devoted to providing monad compositions as a basis for a notion of generalised terms; specifically, we have proven that the \(L\)-fuzzy functors given by the extension principles can be extended to a monad as well as the composition of the \(L\)-fuzzy functors with the term monad.

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