Exponentially-fitted methods on layer-adapted meshes

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Abstract

In this paper, a new derivation of a uniformly-convergent, second-order method for singularly-perturbed, linear ordinary differential equations based on the freezing of the coefficients of the differential equation, and integration of the resulting equations subject to continuity and smoothness conditions at the nodes, is presented. The derivation presented here is compared with others based on Green’s functions, when only advection and diffusion processes are considered when solving the homogeneous equations. In addition, a new method that accounts for advection, diffusion and reaction processes when solving the homogeneous equation is also presented. The two exponentially-fitted techniques presented in the paper are used on layer-adapted meshes which are piecewise uniform and concentrate grid points in the boundary layers, and their results are compared with those obtained with upwind methods in piecewise-uniform meshes. It is shown that standard techniques on piecewise-uniform meshes are less accurate than exponentially-fitted ones, and the accuracy of the latter may not improve by employing layer-adapted piecewise-uniform meshes due to the large change in the step size at the transition points. The paper also presents an exponentially-fitted method for singularly-perturbed, periodic, two-point boundary-value problems of ordinary differential equations.

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1. Introduction

This paper is concerned with the development of exponentially-fitted techniques for singularly-perturbed, linear ordinary differential equations

\[-\epsilon u'' + b(x)u' + c(x)u = f(x), \quad 0 < x < 1,\]  

where \(\epsilon \ll 1\), although the methods presented here are also applicable for finite values of \(\epsilon\).

Exponentially-fitted methods have a long history starting with the uniformly-convergent first-order accurate techniques of Allen and Southwell [1], Ii’in [2] and Scharfetter and Gummel [3] which can be derived by considering the interval \([x_{i-1}, x_i]\), freezing the coefficients of Eq. (1) at \(x_i\), integrating the resulting constant-coefficients ordinary differential equation and imposing the conditions \(u(x_{i-1}) = u_i - 1, u(x_i) = u_i\) and \(u(x_i + 1) = u_i + 1\), to deduce a three-point finite difference equation which can be shown to reduce to a first-order upwind method when \(\frac{|b(x)|}{\epsilon} \rightarrow \infty\) for \(b(x) \neq 0\), where \(h\) denotes the step size. These methods can also be derived in many other manners as indicated by Morton [4], Roos et al. [5], Roos [6] and Ramos [7], although formulations based on the local Green’s function have been frequently used in the literature for their derivation [4,5].

In 1978, El-Mistikawy and Werle [8] proposed an exponentially-fitted method which uses some of the flexibility of compact operator implicit techniques and can be obtained by determining the local Green’s function on the interval \([x_{i-1}, x_i]\) and considering the solution in the disjoint intervals \([x_{i-1}, x_i]\) and \([x_i, x_i + 1]\), when due attention is paid to the approximations that are required to obtain accurate solutions [4,5]. The El-Mistikawy and Werle [8] exponentially-fitted method satisfies an error bound of the form \(Ch^2\), where \(C\) is a constant independent of both \(h\) and \(\epsilon\), \(h\) is the largest step size, and this error exhibits the highest order achievable for a uniform error bound by a three-point scheme [4]. In addition, this scheme is usually deduced by considering only the advection and diffusion operators in Eq. (1), i.e., by considering

\[-\epsilon u'' + b(x)u' = f(x) - c(x)u, \quad 0 < x < 1,\]  

and treating the reaction term \(c(x)u\) as a source.

In this paper, we present in Section 2 a new derivation of the El-Mistikawy and Werle [8] exponentially-fitted method based on partitioning the interval [0,1] into disjoint intervals, freezing the coefficients and right-hand side of
Eq. (2) at $x_L$ and $x_R$ where $x_{i-1} \leq x_L \leq x_i$, $x_i \leq x_R \leq x_{i+1}$, integrating analytically the resulting ordinary differential equations and imposing the conditions $u(x_{i-1}) = u_{i-1}$, $u(x_i) = u_i$, $u(x_{i+1}) = u_{i+1}$ and $u'(x_i^-) = u'(x_i^+)$ that result in a tree-point finite difference method. This new derivation has numerous advantages over those based on the local Green’s function, and these are discussed at some length in Section 2.

In Section 3, we present a generalization of the El-Mistikawy and Werle [8] exponentially-fitted method based on a technique analogous to that employed in Section 2, but which now accounts for advection, diffusion and reaction, i.e., the left-hand side of Eq. (1), when determining the piecewise analytical solutions, whereas that of Section 2 only accounts for advection and diffusion effects when determining the homogeneous solution. Section 3 considers the case $b(x) \neq 0$ and $c(x) \neq 0$, as well as the case $b(x) = 0$ and $c(x) \neq 0$; the first case exhibits a boundary layer of thickness $O(\epsilon)$ at one of the boundary points, whereas the second one exhibits boundary layers at both end points. In both cases, the generalization of the El-Mistikawy and Werle [8] exponentially-fitted method is based on the $C^1$ technique developed by the author [9] for both ordinary differential equations and parabolic partial differential equations [10–13]. In fact, the techniques presented in Sections 2 and 3 can also be used to solve multidimensional singularly-perturbed non-linear advection–reaction–diffusion equations upon linearizing with respect to time and approximating the multidimensional operator by a sequence of one-dimensional ones [14].

The exponentially-fitted techniques described in Sections 2 and 3 can also be applied to study singularly-perturbed, periodic, two-point boundary-value problems such as those considered by Amiraliyev and Duru [15] and Duru [16]. However, in Section 4, we present another exponentially-fitted technique which does not employ a smoothness condition, and prove that this technique reduces to the weak formulation of Amiraliyev and Duru [15]. In that section, we also show that the $C^0$ method [9] can also be written in a more compact form and deduce the finite difference form of the periodic boundary conditions. The term “periodic” used throughout the paper does not correspond to the usual meaning of periodicity in applied mathematics and physics, for the function is periodic but its first-order derivative shows a jump. The formulation presented in Section 4, however, is general enough so that it can also be applied to purely periodic two-point boundary-value problems.

In Section 5, the exponentially-fitted methods described in Sections 2–4 are used on layer-adapted meshes which are piecewise-uniform [5,17,18], and the results are compared with those based on standard upwind techniques on layer-adapted meshes in Section 6 in order to assess the accuracy of exponentially-fitted schemes. A section on conclusions puts an end to the paper.

The El-Mistikawy and Werle [8] method has been previously derived by means of locally-exact methods based on the use of the local Green’s functions \[4,5\]. In this section, this method is derived by using the $C^1$ exponentially-fitted method developed by the author [9]. This derivation is based on the use of only the non-self-adjoint advection–diffusion operator rather than on the use of the advection–reaction–diffusion operator of Eq. (1).

Consider the following singularly-perturbed, linear ordinary differential equation

$$L(u) \equiv -\epsilon u'' + b(x)u' = F(x) \equiv f(x) - c(x)u, \quad 0 < x < 1,$$

subject to, for example,

$$u(0) = a, \quad u(1) = b,$$

where primes denote differentiation with respect to the independent $x$ variable, $a$ and $b$ are constants, and $0 < \epsilon \ll 1$, $c(x) \geq 0$ and $-b(x) \geq b^* > 0$ with $b^*$ equal to a constant.

In order to solve Eq. (3) numerically, the interval $[0,1]$ is divided into $N$ non-overlapping intervals $L \equiv [x_{i-1}, x_i]$ and $R \equiv [x_i, x_{i+1}]$, so that $x_0 = 0$ and $x_N = 1$. In each interval, $L$, Eq. (3) is approximated by

$$-\epsilon u'' + b_L u = F_L, \quad x_{i-1} < x < x_i,$$

where, for example, $b_L = b(x_L)$ and $x_{i-1} \leq x_L \leq x_i$, and solved analytically to yield

$$u = A_L + B_L \exp \left( \frac{b_L}{\epsilon} (x - x_{i-1}) \right) + \frac{F_L}{b_L} (x - x_{i-1}), \quad x_{i-1} < x < x_i,$$

$$u = A_R + B_R \exp \left( \frac{b_R}{\epsilon} (x - x_i) \right) + \frac{F_R}{b_R} (x - x_i), \quad x_i < x < x_{i+1},$$

subject to $u(x_{i-1}) = u_{i-1}$, $u(x_i) = u_i$, $u(x_{i+1}) = u_{i+1}$, and $u'(x_i^-) = u'(x_i^+)$, which imply that

$$B_L = \frac{1}{\exp(\rho^-) - 1} \left( u_i - \frac{F_L h_L}{b_L} u_{i-1} \right), \quad A_L = u_{i-1} - B_L,$$

$$B_L = \frac{1}{\exp(\rho^+) - 1} \left( u_{i+1} - \frac{F_R h_R}{b_R} u_i \right), \quad A_R = u_i - B_R,$$

and

$$\frac{b_L}{\epsilon} \exp(\rho^-) - 1 \left( u_i - u_{i-1} \right) - \frac{b_R}{\epsilon} \exp(\rho^+) - 1 \left( u_{i+1} - u_i \right)$$

$$= \frac{F_R}{b_R} \left( 1 - \frac{\rho^+}{\exp(\rho^+) - 1} \right) + \frac{F_L}{b_L} \left( \frac{\rho^-}{\exp(\rho^-) - 1} - 1 \right),$$

where $\rho^- = \frac{b_L h_L}{\epsilon}$, $\rho^+ = \frac{b_R h_R}{\epsilon}$. 

}\end{quote}
For $h_L = h_R = h$, Eq. (10) can be written as

$$p_i^+ u_{i+1} - (p_i^- + p_i^+) u_i + p_i^+ u_{i+1} = F_R q_i^+ + F_L q_i^-,$$

(11)

where

$$p_i^- = - \frac{b_L}{h} \frac{1}{1 - \exp(-\rho^-)}, \quad p_i^+ = - \frac{b_R}{h} \frac{\exp(-\rho^+)}{1 - \exp(-\rho^+)},$$

$$q_i^- = \frac{1}{2} \left( \frac{1}{1 - \exp(-\rho^-)} - \frac{1}{\rho^-} \right), \quad q_i^+ = \frac{1}{2} \left( - \frac{\exp(-\rho^+)}{1 - \exp(-\rho^+)} + \frac{1}{\rho^+} \right),$$

(13)

where $F_R = \frac{1}{2} (F_i + F_{i+1})$, $F_L = \frac{1}{2} (F_i + F_{i-1})$, $(cu)_R = \frac{1}{2} ((cu)_i + (cu)_{i+1})$ and $(cu)_L = \frac{1}{2} ((cu)_i + (cu)_{i-1})$, so that Eq. (11) can be expressed as

$$\left( p_i^- + \frac{1}{2} c_{i-1} q_i^- \right) u_{i-1} - \left( p_i^- + p_i^+ - \frac{c_i}{2} (q_i^- + q_i^+) \right) u_i + \left( p_i^+ + \frac{1}{2} c_{i+1} q_i^+ \right) u_{i+1}$$

$$= \frac{1}{2} f_{i+1} q_i^+ + \frac{1}{2} f_i (q_i^- + q_i^+) + \frac{1}{2} f_{i-1} q_i^-.$$

(14)

If, in Eqs. (10) and (11), $\rho^- \to \infty$ and $\rho^+ \to \infty$, then $p_i^- \to - \frac{b_L}{h}$, $p_i^+ \to 0$, $q_i^- \to \frac{1}{2}$, $q_i^+ \to 0$, and Eq. (11) becomes

$$\frac{b_L}{h} (y_i - y_{i-1}) = F_L,$$

(15)

which, with $F_L = \frac{1}{2} (F_i + F_{i-1})$, results in a second-order accurate midpoint technique.

Eq. (11) coincides with the one on p. 45 of Roos et al. [5] who employed a (local) Green’s function. It also coincides with Eqs. (3.7) and (3.36) of Morton [4] who states that the El-Mistikawy and Werle’s [8] exponentially-fitted scheme uses some of the flexibility of the operator compact implicit methods. The derivation carried out in this section employs neither the Green’s function nor Hermitian or compact operator formulations; it is a simple derivation based on freezing the coefficients of Eq. (3) and imposing continuity and smoothness conditions on the (piecewise) analytical solutions that result upon integration. This analytical solution has the additional advantage that does not require interpolation to determine the solution at points different from the nodes. In addition, the formulation is not subject to the care that has to be exercised when determining the Green’s functions as indicated next.

The operator $L(u)$ of Eq. (3) is not self-adjoint, but one can determine the local Green’s function as follows. By multiplying Eq. (3) by $v(x)$ and integrating the resulting integrals by parts, it is an easy exercise to show that
\[ \int_{x_{i-1}}^{x_i} vL(u) \, dx = \int_{x_{i-1}}^{x_i} vF(x) \, dx = \int_{x_{i-1}}^{x_i} uL^*(v) \, dx + \epsilon(uv' - u'v)_{i+1} - \epsilon(uv' - u'v)_{i-1} + (buw)_{i+1} - (buw)_{i-1}, \]

(16)

where \( L^*(v) = -\epsilon v'' - (bv)' \).

By making \( L^*(v) = -\epsilon v'' - (bv)' = \delta(x - x_i) \), imposing \( v_{i-1} = v_{i+1} = 0 \), \( v(x_i^-) = v(x_i^+) \) and \( \epsilon(v'(x_i^-) - v'(x_i^+)) = 1 \), one can determine \( v(x) \) in \([x_{i-1}, x_i]\) and \([x_i, x_{i+1}]\), and, therefore, a tridiagonal system of algebraic equations for \( u_i \) through Eq. (16). However, one must exercise great care when determining the local Green’s function, for \( L^*(v) = -\epsilon v'' - (bv)' \) is not a self-adjoint operator which may be written as \( L^*(v) = -\epsilon v'' - bv' - b'v \).

If one were to employ the same piecewise linearization method on \( L^*(v) = -\epsilon v'' - bv' - b'v \) as the one that we used previously in this section, one would not recover the El-Mistikawy and Werle’s \cite{8} exponentially-fitted scheme unless \( L^*(v) = -\epsilon v'' - (bv)' = \delta(x - x_i) \) is integrated analytically to yield

\[ \epsilon v' + bv = C_L, \quad x_{i-1} < x < x_i \]

(17)

and

\[ \epsilon v' + bv = C_R, \quad x_i < x < x_{i+1}, \]

(18)

where \( C_L \) and \( C_R \) are integration constants.

Integration of Eqs. (17) and (18) yields

\[ v = \exp \left( -\int_{x_{i-1}}^{x} \frac{b(x)}{\epsilon} \, dx \right) \left( D_L + \frac{C_L}{\epsilon} \int_{x_{i-1}}^{x} \exp \left( \int_{x_{i-1}}^{x} \frac{b(\eta)}{\epsilon} \, d\eta \right) \, dx \right), \]

(19)

where \( D_L \) and \( D_R \) are integration constants. \( C_L, C_R, D_L \) and \( D_R \) can be determined from the conditions \( v_{i-1} = v_{i+1} = 0 \), \( v(x_i^-) = v(x_i^+) \) and \( \epsilon(v'(x_i^-) - v'(x_i^+)) = 1 \).

It is in Eqs. (19) and (20) were accurate quadrature rules should be used to determine the integrals rather than approximating \( L^*(v) = -\epsilon v'' - (bv)' = 0 \) on \( x_{i-1} < x < x_i \) and \( x_i < x < x_{i+1} \) by \( b'v + b_{Le} = 0 \) and \( b'v + b_{Re} = 0 \), respectively, because these approximations neglect the term \( bv \) compared with the terms kept in the approximation. Furthermore, if these approximations were...
made, one could not obtain Eq. (14), i.e., the El-Mistikawy and Werle [8] method. A similar comment applies if $L'(v) = -\epsilon v'' - (bv)' = 0$ on $x_{i-1} < x < x_i$ and $x_i < x < x_{i+1}$ is approximated by $\epsilon v'' + b_L' v = 0$ and $\epsilon v'' + b_R' v = 0$, respectively, because, in this case, the term $bv'$ is assumed to be smaller than the other two terms maintained in the approximation. Moreover, by defining $B(x) = \int b(x) dx$, one can write Eqs. (19) and (20) as

$$v = \exp \left( -\frac{B(x)}{\epsilon} \frac{d}{dx} \right) \left( D_L + \frac{C_L}{\epsilon} \int_{x_{i-1}}^x \exp \left( \frac{B(\eta)}{\epsilon} \frac{d}{d\eta} \right) d\eta \right), \quad x_{i-1} < x < x_i,$$

and

$$v = \exp \left( -\frac{B(x)}{\epsilon} \frac{d}{dx} \right) \left( D_R + \frac{C_R}{\epsilon} \int_{x_i}^x \exp \left( \frac{B(\eta)}{\epsilon} \frac{d}{d\eta} \right) d\eta \right), \quad x_i < x < x_{i+1},$$

which, by the Abel–Liouville’s theorem, imply that the Wronskian of the two independent solutions of $L^*(v) = 0$ is $W(x) = W_0 \exp \left( -\frac{B(x)}{\epsilon} \right)$ which is a function of $x$ and where $W_0$ is a constant. If the Abel–Liouville theorem were to be applied to approximations that result from the integration of, for example, $\epsilon v'' + b_L' v = 0$ and $\epsilon v'' + b_R' v = 0$, the Wronskian of the corresponding solutions would not coincide with the just derived one. Therefore, once again, the use of local Green’s functions to determine exponentially-fitted methods has to be performed with some care in order not to violate theorems from the theory of ordinary differential equations and obtain approximations which are in accord with these theorems. Moreover, since the exponentially-fitted methods presented in this section and paper are not subject to these drawbacks, they offer clear advantages over those based on the use of either the local Green’s function or compact operator implicit techniques.

As indicated at the beginning of this section, the derivation of the El-Mistikawy and Werle [8] method presented here has only considered the advection–diffusion operator of Eq. (3). In the next section, we present a generalized El-Mistikawy and Werle [8] exponentially-fitted method based on the solution of the advection–reaction–diffusion operator of Eq. (1).


In this section, we consider the singularly-perturbed, linear ordinary differential equation

$$\epsilon u'' + a(x)u' - b(x)u = f(x), \quad 0 < x < 1,$$  \hspace{1cm} (23)
subject to Dirichlet, Neumann or Robin boundary conditions, under the assumptions that $a(x) \geq a^* > 0$ and $b(x) \geq 0$.

The interval $[0, 1]$ is divided into $N$ non-overlapping intervals $L \equiv [x_{i-1}, x_i]$ and $R \equiv [x_i, x_{i+1}]$, so that $x_0 = 0$ and $x_N = 1$. In each interval, $L$, Eq. (23) is approximated by

$$cu'' + a_L u' - b_L u = f_L, \quad x_{i-1} < x < x_i,$$

(24)

where, for example, $b_L = b(x_L)$ and $x_{i-1} \leq x_L \leq x_i$, and solved analytically to yield

$$u = \exp(-\lambda_L (x - x_{i-1}))(A_L \cosh(\kappa_L (x - x_{i-1}))) + B_L \sinh(\kappa_L (x - x_{i-1}))) - \frac{f_L}{b_L}, \quad x_{i-1} < x < x_i,$$

(25)

which, subject to $u(x_{i-1}) = u_{i-1}$ and $u(x_i) = u_i$, yields

$$B_L = \frac{1}{\sinh(\kappa_L h_L)} \left( \left( u_i + \frac{f_L}{b_L} \right) \exp(\lambda_L h_L) - \left( u_{i-1} + \frac{f_L}{b_L} \right) \cosh(\kappa_L h_L) \right)$$

(26)

and

$$A_L = u_{i-1} + \frac{f_L}{b_L},$$

(27)

where $h_L = x_i - x_{i-1}$, $\lambda_L = \frac{a_L}{2\epsilon}$ and $\kappa_L = \left( \lambda_L^2 + \frac{b_L}{\epsilon} \right)^{1/2}$, and $A_L$ and $B_L$ are constants.

In a similar manner, one can easily determine

$$u = \exp(-\lambda_R (x - x_i))(A_R \cosh(\kappa_R (x - x_i))) + B_R \sinh(\kappa_R (x - x_i))) - \frac{f_R}{b_R}, \quad x_i < x < x_{i+1},$$

(28)

which, subject to $u(x_{i+1}) = u_{i+1}$ and $u(x_i) = u_i$, yields

$$B_R = \frac{1}{\sinh(\kappa_R h_R)} \left( \left( u_{i+1} + \frac{f_R}{b_R} \right) \exp(\lambda_R h_R) - \left( u_i + \frac{f_R}{b_R} \right) \cosh(\kappa_R h_R) \right)$$

(29)

and

$$A_R = u_i + \frac{f_R}{b_R},$$

(30)

where $h_R = x_{i+1} - x_i$, $\lambda_R = \frac{a_R}{2\epsilon}$ and $\kappa_R = \left( \lambda_R^2 + \frac{b_R}{\epsilon} \right)^{1/2}$, and $A_R$ and $B_R$ are constants.

Upon imposing the smoothness condition $u'(x_i^-) = u'(x_i^+)$, one can easily deduce
\[
\frac{\kappa_L}{\sinh(\kappa_L h_L)} \exp(-\lambda_L h_L) u_{i-1} - (\kappa_R \coth(\kappa_R h_R)) u_i + \frac{\kappa_L}{\sinh(\kappa_R h_R)} \exp(\lambda_R h_R) u_{i+1}
\]

\[
+ \frac{f_R}{b_R} \left( \kappa_R \coth(\kappa_R h_R) + \lambda_R - \frac{\kappa_R}{\sinh(\kappa_R h_R)} \exp(\lambda_R h_R) \right)
\]

\[
+ \frac{f_L}{b_L} \left( \kappa_L \coth(\kappa_L h_L) - \lambda_L - \frac{\kappa_L}{\sinh(\kappa_L h_L)} \exp(-\lambda_L h_L) \right).
\]

(31)

For the functions \(a(x)\) and \(b(x)\) considered here, there is a boundary layer at \(x = 0\). Moreover, if we assume that \(a_L^2 \gg 4\epsilon b_L\) and \(a_R^2 \gg 4\epsilon b_R\), then \(\lambda \approx \kappa\), and, if \(\lambda h \gg 1\), Eq. (31) is equivalent at leading-order to \(u_{i+1} = u_i\), i.e., a first-order upwind method.

Although the exponentially-fitted methods presented in Sections 2 and 3 have used as examples singularly-perturbed, linear, two-point boundary-value problems, they are also applicable to other linear ordinary differential equations. Moreover, these methods are exact for linear second-order ordinary differential equations which have constant coefficients and constant right-hand sides.

If \(a(x) = 0\) in Eq. (23), then this equation has boundary layers at \(x = 0\) and \(x = 1\) of thickness on \(O(\sqrt{\epsilon})\), \(\lambda_L = \lambda_R = 0\), \(\kappa_L = \left(\frac{b_L}{\epsilon}\right)^{1/2}\) and \(\kappa_R = \left(\frac{b_R}{\epsilon}\right)^{1/2}\), and Eq. (31) becomes

\[
\frac{\kappa_L}{\sinh(\kappa_L h_L)} u_{i-1} - (\kappa_R \coth(\kappa_R h_R) + \kappa_L \coth(\kappa_L h_L)) u_i + \frac{\kappa_R}{\sinh(\kappa_R h_R)} u_{i+1}
\]

\[
= \frac{f_R \kappa_R}{b_R} \tanh\left(\frac{\kappa_R h_R}{2}\right) + \frac{f_L \kappa_L}{b_L} \tanh\left(\frac{\kappa_L h_L}{2}\right).
\]

(32)

Similar comments to the ones made at the end of the previous section regarding the derivation of the method presented in this section by means of the local Green function also apply here, for this function is governed by the non-self-adjoint operator of Eq. (23) which has the following adjoint one \(L^*(v) = \epsilon v'' - (av)' - bv\).

4. Singularly-perturbed periodic boundary-value problems

In this section, we consider the following singularly-perturbed, periodic, two-point boundary-value problem

\[
L(u) \equiv \epsilon^2 u'' + \epsilon a(x) u' - b(x) u = f(x), \quad 0 < x < 1,
\]

(33)
subject to
\[ u(0) = u(1), \quad u'(1) - u'(0) = \frac{A}{\epsilon}, \tag{34} \]
where \( \epsilon \ll 1 \), \( a(x) \geq a^* > 0 \) and \( b(x) \geq b^* > 0 \), and \( A \) is a constant. \( a(x) \), \( b(x) \) and \( f(x) \) are assumed to be sufficiently differentiable, and the solution of Eq. (33) exhibits boundary layers at \( x = 0 \) and \( x = 1 \). Eq. (1) appears in mathematical models of liquid crystals, chemical reactions, control theory, electrical networks, etc. [15,16], and can be studied numerically by means of the exponentially-fitted methods presented in Sections 2 and 3 of this paper.

In this section, we present another exponentially-fitted technique for Eq. (33) and compare it with that presented by Amiraliyev and Duru [15] and Duru [16] who also imposed the conditions \( a(0) = a(1), b(0) = b(1) \) and \( f(0) = f(1); \) these conditions are not imposed in the method considered in this section.

Amiraliyev and Duru [15] and Duru [16] constructed a finite difference scheme for Eq. (33) on uniform meshes by applying a weak formulation to that equation based on the integral (over the whole domain) of the product of this equation times exponential basis functions, \( \phi(x) \), which are the solutions of

\[ \epsilon^2 \phi'' - \epsilon a \phi' - b \phi = 0, \quad \phi(x_{i-1}) = 0, \quad \phi(x_i) = 0, \quad x_{i-1} < x < x_i \tag{35} \]
and

\[ \epsilon^2 \phi'' - \epsilon a \phi' - b \phi = 0, \quad \phi(x_i) = 1, \quad \phi(x_{i-1}) = 0, \quad x_i < x < x_{i+1}. \tag{36} \]

The equations governing these basis functions are, in fact, approximations to those governing the adjoint operator of Eq. (33) which can be written as \( L'(v) = \epsilon^2 - \epsilon (av)' - bv \) and is not-self-adjoint. This adjoint operator can be written as \( L'(v) = \epsilon^2 - \epsilon av' - (ea' + b)v \) which is identical to that of Eqs. (35) and (36), if the term \( \epsilon a' \) is neglected (compared with \( b \)), and is related to the local Green function (cf. Sections 2 and 3). As described in Sections 2 and 3 of this paper, care must be exercised when determining an accurate (local) Green’s function of non-self-adjoint operators.

In this section, we show that the finite difference equations obtained by Amiraliyev and Duru [15] and Duru [16] for singularly-perturbed, periodic boundary-value problems at interior points is identical to the one that can be obtained by employing the \( C^0 \) exponentially-fitted methods developed the author [9] in terms of exponential functions. We also show another formulation of these \( C^0 \) techniques which is much simpler than that based on exponential functions, and compare the boundary conditions obtained with these exponentially-fitted techniques with those obtained by Amiraliyev and Duru [15] and Duru [16].

4.1. Equivalence of weak formulations and \( C^0 \) techniques

Eq. (33) can be approximated in the interval \([x_{i-1}, x_{i+1}]\) by the following constant-coefficients ordinary differential equation
\[ e^2 u'' + e \alpha u' - b_i u = f_i, \quad x_{i-1} < x < x_{i+1}, \]  
which has the analytical solution
\[ u(x) = A_i \exp(\lambda_i^+(x - x_i)) + B_i \exp(\lambda_i^-(x - x_i)) - \frac{f_i}{b_i}, \]  
where \( A_i \) and \( B_i \) are constants, and \( \lambda_i^\pm = -\frac{a_i}{2\pi} \pm \left( \left( \frac{a_i}{2\pi} \right)^2 + \frac{b_i}{\pi} \right)^{1/2}. \)

The constants \( A_i \) and \( B_i \) can be determined from the conditions \( u(x_{i-1}) = u_{i-1}, u(x_i) = u_i \) and \( u(x_{i+1}) = u_{i+1} \) which yield
\[ B_i = \frac{1}{T_L} \left( u_{i-1} + \frac{f_i}{b_i} - \left( u_i + \frac{f_i}{b_i} \right) \exp(-\lambda_i^+ h_L) \right) \]
\[ = \frac{1}{T_R} \left( u_{i+1} + \frac{f_i}{b_i} - \left( u_i + \frac{f_i}{b_i} \right) \exp(\lambda_i^+ h_R) \right) \]  
and
\[ A_i = u_i + \frac{f_i}{b_i} - B_i, \]
where \( T_L = \exp(-\lambda_i^- h_L) - \exp(-\lambda_i^+ h_L) \) and \( T_R = \exp(\lambda_i^- h_R) - \exp(\lambda_i^+ h_R) \).

The three-point finite difference Eq. (39) is valid at interior points and, for \( h_L = h_R = h \), may be written as
\[ u_{i-1} + \frac{f_i}{b_i} - (\exp(-\lambda_i^+ h) + \exp(-\lambda_i^- h)) \left( u_i + \frac{f_i}{b_i} \right) + \exp(-\lambda_i^+ + \lambda_i^-) h \left( u_{i+1} + \frac{f_i}{b_i} \right) = 0, \]  
which, after some lengthy algebra and the use of some relationships amongst hyperbolic and exponential functions, can, in turn, be written as
\[ e^2 \frac{\theta_i}{h^2} \delta^2 u_i + e \frac{\psi_i a_i}{2h} \mu u_i - b_i u_i = f_i, \]  
where \( \delta^2 u_i = u_{i+1} - 2u_i + u_{i-1} \) and \( \mu u_i = u_{i+1} - u_{i-1} \) are central difference operators, and
\[ \theta_i = -\frac{b_i h^2}{2e^2 T_i} \left( 1 + \exp((\lambda_i^+ + \lambda_i^-) h) \right) \]
\[ = -\frac{b_i h^2}{4e^2} \left( 1 + \coth \left( \frac{\lambda_i^+ h}{2} \right) \coth \left( \frac{\lambda_i^- h}{2} \right) \right), \]
\[ \psi_i = \frac{b_i h}{e \alpha_i T_i} \left( -1 + \exp((\lambda_i^+ + \lambda_i^-)h) \right) \]
\[ = \frac{b_i h}{2 e \alpha_i} \left( \coth \left( \frac{\lambda_i^+ h}{2} \right) + \coth \left( \frac{\lambda_i^- h}{2} \right) \right), \]  
(44)

which coincide with Eqs. (3.5) and (3.6) of Amiraliyev and Duru [15] after some typographical errors are corrected in their Eq. (3.6), and where

\[ T_i = \exp((\lambda_i^+ + \lambda_i^-)h) - \exp(\lambda_i^+ h) - \exp(\lambda_i^- h) + 1. \]

Eqs. (43) and (44) provide fitting factors which have been determined without taking into consideration the boundary layers and their locations, and their derivation should be contrasted with that based on the use of the inner or boundary layer asymptotic expansions provided by other authors [20].

4.2. Another form of the \( C^0 \) exponentially-fitted technique

In this section, we present a new form of the \( C^0 \) exponentially-fitted technique [9] which is somewhat much simpler than the one presented in the previous section. We first define \( \lambda_i = \frac{\nu}{2 \varepsilon} \) and \( \kappa_i = \left( \left( \frac{\nu}{2 \varepsilon} \right)^2 + \frac{b_i}{e \alpha_i} \right) \), so that the solution to Eq. (37) can be written as

\[ u = \exp(-\lambda_i(x - x_i))(A_i \cosh(\kappa_i(x - x_i)) + B_i \sinh(\kappa_i(x - x_i))) - \frac{f_i}{b_i}, \]
(45)

where the conditions \( u(x_i - 1) = u_{i-1}, u(x_i) = u_i \) and \( u(x_i + 1) = u_{i+1} \) result in

\[ B_i = \frac{1}{\sinh(\kappa_i h_L)} \left( \cosh(\kappa_i h_L) \left( u_i + \frac{f_i}{b_i} \right) - \left( u_{i-1} + \frac{f_i}{b_i} \right) \exp(-\lambda_i h_L) \right) \]
\[ = \frac{1}{\sinh(\kappa_i h_R)} \left( -\cosh(\kappa_i h_R) \left( u_i + \frac{f_i}{b_i} \right) + \left( u_{i+1} + \frac{f_i}{b_i} \right) \exp(\lambda_i h_R) \right), \]
(46)

\[ A_i = u_i + \frac{f_i}{b_i}, \]
(47)

and the three-point finite difference equation

\[ -\frac{\exp(-\lambda_i h_L)}{\sinh(\kappa_i h_L)} \left( u_{i-1} + \frac{f_i}{b_i} \right) + (\coth(\kappa_i h_L) + \coth(\kappa_i h_R)) \left( u_i + \frac{f_i}{b_i} \right) \]
\[ -\frac{\exp(\lambda_i h_R)}{\sinh(\kappa_i h_R)} \left( u_{i+1} + \frac{f_i}{b_i} \right) = 0, \]
(48)

which is valid at the interior points, i.e., at \( i = 2, 3, \ldots, NP - 1 \), where \( NP \) is the number of grid points.

Eq. (47) is to be solved subject to the finite difference form of Eq. (34) which yields
and

\[
- \frac{\kappa_{NM}}{\sinh(\kappa_{NM} h_{NP})} \exp(-\lambda_{NM} h_{NP}) \left( u_{NM} + \frac{f_{NM}}{b_{NM}} \right) + (\kappa_{NM} \cosh(\kappa_{NM} h_{NP})) \\
- \lambda_{NM} \sin(\kappa_{NM} h_{NP})) \left( u_{NP} + \frac{f_{NM}}{b_{NM}} \right) - \frac{\kappa_2}{\sinh(\kappa_2 h_2)} \exp(\lambda_2 h_2) \left( u_2 + \frac{f_2}{b_2} \right) \\
+ (\kappa_2 \cosh(\kappa_2 h_2) + \lambda_2 \sin(\kappa_2 h_2)) \left( u_1 + \frac{f_2}{b_2} \right) = \frac{A}{\epsilon},
\]

(50)

where \( NM \equiv NP - 1 \).

Eqs. (47), (49) and (50) yield the following system of linear algebraic equations

\[
C_iu_{i-1} + D_iu_i + E_iu_{i+1} = F_i, \quad i = 2, 3, \ldots, NM - 1,
\]

\[
C_{NM}u_{NM1} + D_{NM}u_{NM} + E_{NM1}u_1 = F_{NM1},
\]

\[
C_1u_1 + D_1u_2 + E_1u_{NM} = F_1,
\]

(51)

for \( u_i, i = 1, 2, \ldots, NM \) which correspond to Eq. (47), Eq. (47) applied at \( i = NM \) with \( u_1 = u_{NP} \) and Eq. (50), respectively, and where \( NM1 = NM - 1 \) and the coefficients of the finite difference equations can be easily determined from Eqs. (47), (49) and (50).

It must be pointed out that Eq. (50) does to coincide with Eq. (3.14) of Amiraliev and Duru [15] and Duru [16] because their flux condition only includes the value of \( f_1 \) whereas Eq. (50) includes \( f_2 \) and \( f_{NM} \). In addition, Eq. (50) was derived from the piecewise analytical solution given by Eq. (45).

Eq. (51) can be solved by Gauss elimination or the factorization procedure of Samarskii [23].

5. Exponentially-fitted methods on layer-adapted meshes

In this section, we use the exponentially-fitted methods developed in Sections 2–4 on piecewise uniform meshes of the Shishkin type [5,17,18]. For Eq. (23) with \( a(x) \geq a^* > 0 \) and \( b(x) \geq b^* > 0 \) that exhibits a boundary layer at \( x = 0 \), the interval \([0, 1]\) is divided into the subintervals \([0, \sigma]\) and \([\sigma, 1]\) where the transition coordinate \( \sigma = \frac{a}{2} \ln(N) \), with \( \frac{N}{2} \) intervals in \([0, \sigma]\) and \([\sigma, 1]\) each, so that the step size in these interval is equal to \( \frac{2\sigma}{N} \) and \( \frac{2(1-\sigma)}{N} \), respectively. For Eq. (23) with \( a(x) = 0 \) and \( b(x) \geq b^* > 0 \), there are boundary layers at \( x = 0 \) and \( x = 1 \), and the interval \([0, 1]\) is divided into the subintervals \([0, \sigma]\), \([\sigma, 1 - \sigma]\), and \([1 - \sigma, 1]\) with \( \frac{N}{3} \) intervals in each, so that the grid spacing in these intervals is \( \frac{2\sigma}{N}, \frac{2(1-2\sigma)}{N} \) and \( \frac{2\sigma}{N} \), respectively, where now \( \sigma = \frac{a}{2} \ln(N) \) because the boundary layers at \( x = 0 \) and \( x = 1 \) have a thickness on \( O(\sqrt{\epsilon}) \).
The exponentially-fitted methods presented in this paper can use other adaptive techniques than the layer-adapted one described above because they do provide piecewise analytical solutions which depend in an exponential manner on $\lambda$ or $\kappa$. Therefore, depending on the variation of the solution or the absolute value of the slope of the solution on each interval, one may select the step size so that either the variation or the slope of the solution does not exceed a user's specified threshold value. This mesh adaption technique is, therefore, iterative, for both the mesh and the solution are coupled. Moreover, the accuracy of the exponentially-fitted methods presented in this paper can be improved by using Richardson's extrapolation, the modified-equation approach or defect-correction techniques, at least, where the mesh is piecewise-uniform [21,22].

It must be stated that the use of specially-designed layer-adapted meshes of the Shishkin type requires a knowledge of the location of the boundary layers as well as of the thickness of these layers [5,17,18]. If the location of these layers were not known, one could perform calculations to determine where these layers are located by means of, for example, the exponentially-fitted methods presented in this paper and then refine the grid around these locations in order to resolve them accurately. The fact that, in order to develop uniformly-convergent methods (on the perturbation parameter), special layer-adapted meshes are required, places some limitations on the use of these techniques in engineering problems.

6. Results

Before presenting the results, it is convenient to emphasize that Eq. (23) was also solved on the piecewise-uniform mesh described in the previous section using central differences for the diffusion terms and upwind differences for the advective ones, so that its discretization can be written as

$$\frac{\epsilon}{\Delta h} \left( \frac{u_{i+1} - u_i}{h_R} - \frac{u_i - u_{i-1}}{h_L} \right) + a_i \frac{u_{i+1} - u_i}{h_R} - b_i u_i = f_i,$$

(52)

where $h = x_{i+1} - x_{i-1}$. Consider the following linear equation [19]

$$\epsilon y'' - y = \cos^2(\pi x) + 2\epsilon \pi^2 \cos(2\pi x),$$

(53)

subject to $y(0) = y(1) = 0$ which has the exact solution

$$y_e(x) = \left[ \exp\left(\frac{1-x}{\sqrt{\epsilon}}\right) + \exp\left(-x\sqrt{\epsilon}\right) \right] / \left[ 1 + \exp\left(-1/\sqrt{\epsilon}\right) \right] - \cos^2(\pi x),$$

(54)

and exhibits boundary layers at $x = 0$ and $x = 1$ of thickness on $O(\epsilon)$.

Eq. (53) was solved by means of the exponentially-fitted method presented in Sections 2 and 3 (superscript ef) and the standard difference technique of
Eq. (52) (superscript sd) on layer-adapted meshes of the Shishkin type (subscript S) and equally-spaced grids (no subscript). The exponentially-fitted method of Sections 2 and 3 was also used on a layer-adapted mesh that divides the interval $[0,1]$ into the subintervals $[0, \sigma^*]$, $[\sigma^*, 1 - \sigma^*]$ and $[1 - \sigma^*, 1]$, $\sigma^* = 10 \sqrt{\epsilon}$ and the same number of grid points were used in these intervals where equally-spaced grids are employed; the resulting method was been labelled with the subscript BL and concentrates a large number of grid points in the boundary layer, the thickness of which has been set to $\sigma^*$.

It must be noted that the two layer-adapted meshes employed in this study use piecewise-uniform meshes and, as a consequence, the step size undergoes a large change at the transition points $\sigma$ and $1 - \sigma$, and $\sigma^*$ and $1 - \sigma^*$. These abrupt changes in the step size are expected to have a large influence on the accuracy of both standard finite difference and exponentially-fitted methods, especially, for small values of the perturbation parameter. It must also be noted that a smoother layer-adapted mesh can be developed by considering stretched grids in $[\sigma, 1 - \sigma]$ so that the grid spacing increases in a smooth manner from $\sigma$ to, say, $x = \frac{1}{2}$, and then decreases in an also smooth manner from this point up to $1 - \sigma$. Examples of smooth meshes include those based on exponential functions or geometric progressions. Alternatively, stretched meshes can be developed by equidistributing the arc-length of the solution and, therefore, these meshes have to be determined together with the solution of the ordinary differential equation in an iterative manner.

For Eq. (53), the results presented in Table 1 indicate that the exponentially-fitted method presented in Sections 2 and 3 on equally-spaced grids is more accurate than the standard finite difference technique on both equally-spaced grids and layer-adapted meshes of the Shishkin type. The accuracy of standard finite difference methods on Shishkin meshes is, in general, higher than when these techniques are employed on equally-spaced meshes except for very small values of the perturbation parameter and not a very large number of grid points, for which the largest errors have been observed at the transition points where the mesh undergoes a large change in step size. This result has been corroborated by considering that the boundary layer is well-resolved on a Shishkin mesh, especially, for $N = 1024$, and indicates that, despite the uniform convergence on the perturbation parameter of standard finite difference methods on layer-adapted meshes of the Shishkin type and the good resolution of the boundary layers, the accuracy of standard finite difference methods may be poor due to the large change in step size at the transition points.

The effects of the large step size change at the transition points can also be observed in Table 2 which corresponds to results obtained with the exponentially-fitted method presented in Sections 2 and 3. This table shows that, for very small values of the perturbation parameter, the second layer-adapted mesh employed in this paper, i.e., that based on $\sigma^*$, yields more accurate results than when the same technique is applied on a Shishkin mesh. Moreover, a
comparison between the results presented in Tables 1 and 2 clearly indicates that the exponentially-fitted method on equally-spaced grids yields more accurate results than when the same technique is applied on layer-adapted meshes. Once again, this is a consequence of the large errors that occur at the transition points of piecewise-uniform layer-adapted meshes, even though the boundary layer is resolved accurately on those meshes. Although not shown here, the

| Table 1 | Absolute errors ($E = \max |y'(x_i) - y'(x_i)|$) for $\epsilon y'' - y = \cos^2(\pi x) + 2\epsilon \pi^2 \cos(2\pi x)$, $y(0) = y(1) = 0$ for different number of grid points $NP = N + 1$ |
|--------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\epsilon$ | $N$ | $E^f$  | $E^d$  | $E^s$  | $E^d$  | $E^d$  |
| $\epsilon = 1$ | 16 | 1.2215E–2 | 1.8259E–3 | 1.9739E–3 | 1.9739E–5 | 1.9739E–7 |
| $\epsilon = 10^{-2}$ | 64 | 7.5809E–3 | 6.4350E–3 | 3.5709E–4 | 1.9738E–5 | 1.9739E–7 |
| $\epsilon = 10^{-6}$ | 1024 | 1.1718E–5 | 1.1064E–5 | 5.9509E–6 | 5.2293E–6 | 1.9739E–7 |
| $\epsilon = 10^{-8}$ | 16 | 1.8186 | 1.4133E–1 | 2.4277E–2 | 2.7483E–4 | 2.7496E–6 |
| $\epsilon = 10^{-2}$ | 64 | 1.8076 | 1.4342E–1 | 2.9849E–2 | 4.0832E–3 | 4.1163E–5 |
| $\epsilon = 10^{-4}$ | 256 | 1.8070 | 1.4355E–1 | 3.6935E–3 | 3.8041E–2 | 6.5471E–4 |
| $\epsilon = 10^{-6}$ | 1024 | 1.8069 | 1.4355E–1 | 2.1048E–3 | 3.5577E–2 | 2.6080E–3 |
| $\epsilon = 10^{-8}$ | 16 | 1.4171E–1 | 4.0616E–2 | 5.4132E–2 | 5.5414E–2 |
| $\epsilon = 10^{-2}$ | 64 | 1.4333E–1 | 1.9550E–3 | 1.3831E–2 | 1.5017E–2 |
| $\epsilon = 10^{-4}$ | 256 | 1.4351E–1 | 1.9644E–3 | 2.7630E–3 | 3.7899E–3 |
| $\epsilon = 10^{-6}$ | 1024 | 1.4357E–1 | 1.9657E–3 | 9.6690E–4 | 1.8200E–3 |

| Table 2 | Absolute errors ($E = \max |y'(x_i) - y'(x_i)|$) for $\epsilon y'' - y = \cos^2(\pi x) + 2\epsilon \pi^2 \cos(2\pi x)$, $y(0) = y(1) = 0$ for different number of grid points $NP = N + 1$ |
|--------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\epsilon$ | $N$ | $E^f$  | $E^d$  | $E^d$  | $E^d$  | $E^d$  |
| $\epsilon = 10^{-2}$ | 16 | 1.3921E–1 | 1.7726E–3 | 1.7830E–5 |
| $\epsilon = 10^{-4}$ | 64 | 1.4299E–1 | 2.3100E–3 | 3.0155E–5 | 5.2595E–7 |
| $\epsilon = 10^{-6}$ | 256 | 1.4353E–1 | 1.9644E–3 | 2.7630E–3 | 4.6693E–7 |
| $\epsilon = 10^{-8}$ | 1024 | 1.4366E–1 | 1.9413E–3 | 4.7233E–5 | 1.0952E–5 |
| $\epsilon = 10^{-2}$ | 16 | 1.3921E–1 | 1.7726E–3 | 1.7830E–5 | 5.2595E–7 |
| $\epsilon = 10^{-4}$ | 64 | 1.4299E–1 | 2.3100E–3 | 3.0155E–5 | 5.2595E–7 |
| $\epsilon = 10^{-6}$ | 256 | 1.4353E–1 | 1.9644E–3 | 2.7630E–3 | 4.6693E–7 |
| $\epsilon = 10^{-8}$ | 1024 | 1.4366E–1 | 1.9413E–3 | 4.7233E–5 | 1.0952E–5 |

comparison between the results presented in Tables 1 and 2 clearly indicates that the exponentially-fitted method on equally-spaced grids yields more accurate results than when the same technique is applied on layer-adapted meshes. Once again, this is a consequence of the large errors that occur at the transition points of piecewise-uniform layer-adapted meshes, even though the boundary layer is resolved accurately on those meshes. Although not shown here, the
accuracy of the results on layer-adapted meshes improves as the number of intervals, \( N \), is increased and by using smaller grid spacings in the outer region, i.e., in \([\sigma, 1 - \sigma]\). For example, calculations performed on a Shishkin mesh with \( N_1, 4N_1 \) and \( N_1 \) intervals on \([0, \sigma^*], [\sigma^*, 1 - \sigma^*] \) and \([1 - \sigma^*, 1] \), \( \sigma^* = 10\sqrt{\epsilon} \), respectively, are more accurate than those reported in Tables 1 and 2.

The results presented in Table 1 also indicate that, for the same number of grid points, standard finite difference discretizations of singularly-perturbed, linear ordinary differential equations produce larger errors than exponentially-fitted techniques even when the former employ layer-adapted meshes, because the largest errors of these standard techniques occur in either the boundary layers at \( x = 0 \) and \( x = 1 \) or at the mesh transition points. Note that Eq. (53) is invariant under the translation \( x \to x - \frac{1}{2} \) and, therefore, the numerical solution should be symmetric with respect to \( x = \frac{1}{2} \). However, it was found that the standard finite difference method on layer-adapted meshes does not provide symmetric solutions for all the values of the perturbation parameter considered in this paper.

The exponentially-fitted method of Section 4 has been applied to Eq. (33) with \( a(x) = 2(\sin(2\pi x) + 1.5), A = 6, b(x) = 6.75 - \sin^2(2\pi x) - 3\sin(2\pi x) - 2\epsilon\pi \cos(2\pi x) \) and \( f(x) = -2(\epsilon\pi^2 \cos(2\pi x) - \epsilon\pi^2(\sin(2\pi x) + 1.5)\sin(2\pi x) + (6.75 - \sin^2(2\pi x) - 3\sin(2\pi x) - 2\epsilon\pi \cos(2\pi x))\sin^2(\pi x) \) which has the following exact solution [15]

\[
\begin{align*}
  u(x) &= \frac{e^{\cos(2\pi x) - 1 / 2\epsilon}}{\cosh(\frac{x}{2\epsilon}) - \cosh(\frac{1}{2\epsilon})} \left( e^{\frac{3(1 - x)}{\epsilon}} \sinh\left( \frac{3(1 - x)}{\epsilon} \right) + e^{\frac{3(1 - x)}{\epsilon}} \sinh\left( \frac{3x}{\epsilon} \right) \right) \\
  &\quad - \sin^2(\pi x),
\end{align*}
\]

and exhibits boundary layers at \( x = 0 \) and \( x = 1 \). In Table 3, we show a comparison between the results obtained with the exponentially-fitted technique presented in Section 4 and the weak formulation of Amiraliyev and Duru [15] which have been labelled with the superscripts \( ef \) and \( wf \), respectively, on equally-spaced meshes.

Table 3 shows that there are very few differences between the results of the weak variational formulation of Amiraliyev and Duru [15] and the exponentially-fitted method of Section 5, despite the fact that these two formulations result in different finite difference equations at the boundaries.

A comparison amongst Tables 1–3 indicates that the accuracy of exponentially-fitted techniques depends on the boundary conditions, for, as indicated in Table 1, these techniques have a high accuracy for Dirichlet boundary conditions. However, the accuracy of these techniques is lower when the boundary conditions are of the Neumann or Robin types than for those of the Dirichlet type. This is perhaps to be expected because the boundary conditions of the Neumann and Robin type are imposed based on the (approximate) piecewise analytical solutions on the leftmost and rightmost intervals, and these solutions
are obtained by freezing the coefficients of the differential equation at either the
next grid point or the midpoint closest to the boundaries of the domain for
$C^0$ and $C^1$ exponentially-fitted techniques, respectively.

### 7. Conclusions

Exponentially-fitted methods for problems involving boundary layers in linear ordinary differential equations have been developed based on freezing the coefficients of the equation, analytical integration and imposition of continuity and smoothness conditions, taking into consideration advection and diffusion, and advection, diffusion and reaction processes when solving the homogeneous equations. These methods are compared with other locally-exact techniques based on the local Green’s function, and it has been shown that the one must exercise great care when determining an accurate numerical (local) Green’s function for non-self-adjoint operators.

The exponentially-fitted techniques presented here have also been implemented on layer-adapted meshes which concentrate a large number of grid points in the boundary layer, and it has been shown that, for the same number of grid points, exponentially-fitted techniques even on equally-space meshes are more accurate than upwind schemes on piecewise-uniform layer-adapted meshes. Moreover, it has also been shown that exponentially-fitted methods on layer-adapted meshes may be less accurate than the same techniques on equally-spaced grids due to the loss of accuracy at the mesh transition points.

For singularly-perturbed, periodic, two-point boundary-value problems, an exponentially-fitted technique has been derived, and it has been shown that this technique provides the same finite difference equations at interior points and

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introduces more coupling through the boundary conditions than those based on a weak formulation. Despite these differences, the accuracy of the exponentially-fitted method is about the same as that of the weak formulation.

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