Piecewise-linearized methods for oscillators with limit cycles

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Abstract

A piecewise linearization method based on the linearization of nonlinear ordinary differential equations in small intervals, that provides piecewise analytical solutions in each interval and smooth solutions everywhere, is developed for the study of the limit cycles of smooth and non-smooth, conservative and non-conservative, nonlinear oscillators. It is shown that this method provides nonlinear maps for the displacement and velocity which depend on the previous values through the nonlinearity and its partial derivatives with respect to time, displacement and velocity, and yields non-standard finite difference formulae. It is also shown by means of five examples that the piecewise linearization method presented here is more robust and yields more accurate (in terms of displacement, energy and frequency) solutions than the harmonic balance procedure, the method of slowly varying amplitude and phase, and other non-standard finite difference equations.

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1. Introduction

Nonlinear oscillators with smooth or non-smooth nonlinearities have been of great interest for many years [1–5] because they play an important role in dynamics, electronics, etc. In the past, nonlinear single degree-of-freedom oscillators with limit cycles have been analyzed asymptotically by means of perturbation methods such as the harmonic balance procedure [6–8], the Krylov–Bogoliubov–Mitropolski (KBM) or slowly varying amplitude and phase technique [6,7,9–11], etc., which do require the presence of a small parameter and are only valid for small values of the perturbation parameter. The application of either of these techniques has allowed to determine the first-order approximation to a variety of nonlinear oscillators with limit cycles such as the Duffing and van der Pol oscillators. However, the cumbersome algebra required by KBM to obtain second- or higher-order approximations to the limit cycle has limited most of the asymptotic studies performed with this method to a first-order approximation.

Recently, there have been attempts to use a hybrid linearization/harmonic balance method for the analysis of nonlinear oscillators. For example, Mickens [12] proposed a combination of equivalent linearization and averaging techniques. Other authors have applied first linearization and then harmonic balance techniques to the analysis of nonlinear oscillators [13]. These techniques have the advantage that they result in simple algebraic equations, instead of the complicated nonlinear ones that arise upon the application of the harmonic balance method.

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Combined linearization/perturbation methods for perturbed smooth dynamical systems first linearize the ordinary differential equation by either a straightforward linearization or minimization of the difference between the nonlinear terms and the linear ones that approximate them [14], and are approximation techniques that are usually limited to small values of the perturbation parameter.

On the other hand, the application of numerical methods to the calculation of limit cycles of smooth and non-smooth oscillators has faced problems associated with accuracy and conservation; the latter is of paramount importance in conservative single degree-of-freedom systems. In fact, it is well known that there are some numerical methods such as those of the Runge-Kutta family that are non-conservative and, therefore, are not applicable to conservative oscillators. In addition, it is also known that some numerical techniques do require very small time steps to obtain accurate answers. This state of affairs has resulted in the development of specific methods for specific applications with the consequent loss of generality. For example, Mickens [15] has proposed non-standard finite difference techniques for the study of nonlinear oscillators, and applied them to a large variety of smooth and non-smooth oscillators including those with fractional-power nonlinearities [16–19]. As stated before, non-standard methods have to be developed in a case-by-case manner that accounts not only for the rules established by Mickens [15] but also for the symmetries of the ordinary differential equations and, when applicable, the existence of the Hamiltonian. Compared with the non-standard finite difference schemes of Mickens [15] that have to be developed in a case-by-case manner, the piecewise linearization method proposed here provides a unified framework for the analysis of nonlinear oscillators.

In this paper, we present a piecewise linearization method for the study of smooth and non-smooth, single degree-of-freedom oscillators, which is based on the piecewise linearization of the governing nonlinear, second-order ordinary differential equation in time intervals, provides piecewise analytical solutions in each of these intervals, and yields smooth solutions everywhere on account of the continuity and smoothness requirements imposed at the end points of these intervals. This piecewise linearization method provides second-order, linear ordinary differential equations in each interval, and can be written as a linear map that yields the values of the displacement and velocity at time level \( n + 1 \) as functions of the corresponding values and the values of the nonlinearity and its partial derivatives with respect to time, displacement and velocity at the previous time level, i.e., at \( n \). As a consequence of both the piecewise linearization and the imposition of continuity and smoothness requirements at the end of each interval, the method also results in non-standard finite difference formulae and may use variable step sizes.

The paper has been arranged as follows. In Section 2, the piecewise linearization method is formulated for smooth and non-smooth single degree-of-freedom, nonlinear oscillators. In that section, we also show that, upon imposing continuity and smooth requirements at the end of each interval, the method results in non-standard finite difference formulae that depend on the derivatives of the nonlinearities with respect to time, displacement and velocity. The method is applied to a variety of conservative and non-conservative, smooth and non-smooth oscillators in Section 4, where its results are compared with available exact, asymptotic and numerical solutions in terms of the predicted displacement, velocity, energy and angular frequency. The paper ends with a summary of the major findings.

2. Formulation

In this paper, we shall be concerned with the following single degree-of-freedom equation:

\[
\ddot{x} = f(t, x, \dot{x}), \quad t > 0, \tag{1}
\]

subject to \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \), where \( f \) is continuous and differentiable with respect to \( t \), and continuous but not necessarily differentiable with respect to \( x \) and/or \( \dot{x} \), e.g., \( f(t, x, \dot{x}) = Ax^{\rho} \) where \( p \) is a rational number such that \( 0 < p < 1 \) and \( A \) is a constant.

Eq. (1) cannot, in general, be integrated analytically, and, therefore, its solution must be determined numerically. In this paper, we first decompose the interval of integration \( 0 \leq t \leq T (T > 0) \) into non-overlapping intervals \( t_n \leq t < t_{n+1} \) where \( n \) is a natural number including zero such that \( t_0 = 0 \).

If, in each interval, Eq. (1) is linearized with respect to the previous time level, one can easily obtain the following second-order, linear, ordinary differential equation

\[
\ddot{x} = f_n + T_n(t - t_n) + J_n(x - x_n) + H_n(\dot{x} - \dot{x}_n), \quad t_n \leq t < t_{n+1}, \tag{2}
\]

provided that \( T_n, J_n \) and \( H_n \) are defined at \( t_n \), where \( f_n = f(t_n, x_n, \dot{x}_n) \), \( T_n = \frac{df}{dt}(t_n, x_n, \dot{x}_n) \), \( J_n = \frac{df}{dx}(t_n, x_n, \dot{x}_n) \) and \( H_n = \frac{df}{d\dot{x}}(t_n, x_n, \dot{x}_n) \), and the right-hand-side of Eq. (1) has been linearized with respect to its three arguments. Eq. (2) is subject to \( x(t_0) = x_n \) and \( \dot{x}(t_0) = \dot{x}_n \).

If \( J_n \) is not defined, then Eq. (1) is only linearized with respect to \( t \) and \( \dot{x} \) provided that \( H_n \) is defined and, in this case, Eq. (2) applies without the third term in its right-hand side. Analogously, if \( H_n \) is not defined, then Eq. (1) is only lin-
earized with respect to $t$ and $x$ provided that $J_n$ is defined and, in this case, Eq. (2) applies without the fourth term in its right-hand side. If both $J_n$ and $H_n$ are not defined, then Eq. (1) is only linearized with respect to $t$ and, in this case, Eq. (2) applies without the third and fourth terms in its right-hand side. Since the three possibilities just described are all included in Eq. (2), i.e., they correspond to setting $H_n = 0$, $J_n = 0$ and $J_n = H_n = 0$, respectively, in that equation, here we shall only consider Eq. (2) while keeping in mind the above comments.

Eq. (2) can be solved analytically in each interval $t_n \leq t < t_{n+1}$ and the form of its solution depends on the roots of the characteristic equation $\lambda^2 - H_n \lambda - J_n = 0$ and on the values of $H_n$ and $J_n$. For example, if $\frac{H_n}{J_n} + J_n < 0$ and $J_n \neq 0$, then the solution to Eq. (2) can be written as

$$x(t) = \exp(\mu_n(t - t_n))(A_n \cos(\Omega_n(t - t_n)) + B_n \sin(\Omega_n(t - t_n))) + x_n + \beta_n(t - t_n), \quad (3)$$

where $\mu_n = \frac{1}{4}H_n$, $\Omega_n^2 = -\left(\frac{H_n}{J_n} + J_n\right)$, $\beta_n = -\frac{\mu_n}{\Omega_n}$, $\gamma_n = -\frac{\mu_n}{\Omega_n}$, $\beta_n = -\frac{\gamma_n}{\Omega_n}$, $P_n = f_n - J_n x_n - H_n \dot{x}_n$, and the values of $A_n$ and $B_n$ can be easily determined from the conditions $x(t_n) = x_n$ and $\dot{x}(t_n) = \dot{x}_n$ as

$$A_n = x_n - \gamma_n, \quad (4)$$
$$B_n = \frac{1}{\Omega_n}(\dot{x}_n - A_n \mu_n - \beta_n). \quad (5)$$

Eq. (3) yields the following explicit finite difference expressions for the conditions on the discriminant and $J_n$ considered above

$$x_{n+1} = x(t_{n+1}) = \exp(\mu_n k_n)(A_n \cos(\Omega_n k_n) + B_n \sin(\Omega_n k_n)) + x_n + \beta_n k_n, \quad (6)$$
$$\dot{x}_{n+1} = \dot{x}(t_{n+1}) = \exp(\mu_n k_n)((A_n \mu_n + B_n \Omega_n) \cos(\Omega_n k_n) + (B_n \mu_n - A_n \Omega_n) \sin(\Omega_n k_n)) + \beta_n, \quad (7)$$

where $k_n = t_{n+1} - t_n$ is the (possibly variable) time step.

Analytical solutions to Eq. (2) for other values of the discriminant, $J_n$ and $H_n$ can be found in any good textbook on linear ordinary differential equations, e.g., [20], and are not reported here.

In this paper, we shall refer to the finite difference formulae of Eq. (2) as the $L$ method (or simply L). This method provides the exact solution to Eq. (1) (in the absence of round-off errors) if $f(t, x, \dot{x})$ is linear function of its three arguments, i.e., if $f(t, x, \dot{x}) = b + c x + d \dot{x} + g \ddot{x}$, where $b$, $c$, $d$ and $g$ are constants. Moreover, the piecewise linearization method presented here results in non-standard finite difference equations, i.e., Eqs. (6) and (7), which are different from those obtained by Mickens [15].

3. Results

In this section, we present some sample results obtained with the piecewise linearization method presented in Section 2 for some smooth and non-smooth, conservative and non-conservative nonlinear oscillators. The accuracy of the piecewise linearization method presented here is assessed by comparing its numerical solutions with analytical or asymptotic ones when available.

Although the piecewise linearization method presented in this paper can use a variable time step that may be adapted to the evolution of the solution, all the results presented here have been obtained with a constant time step, i.e., $k_n = k$.

**Example 1.** This example corresponds to the nonlinear oscillator governed by

$$\ddot{x} + x^3 = c(1 - x^2)\dot{x}, \quad (8)$$
$$x(0) = 0.5, \quad \dot{x}(0) = 0, \quad (9)$$

where $c = 0.1$.

Eq. (9) has a stable limit cycle that may be analyzed by means of the Krylov–Bogoliubov–Mitropolski technique, also known as the method of slowly varying amplitude and phase; it can also be analyzed by means of the harmonic balance method and has a limit cycle as the one exhibited in Fig. 1. This figure shows that, after an initial transient, both the displacement and velocity tend to a periodic behavior whereas the energy, i.e., $E(t) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^2$, oscillates in a periodic fashion. The angular frequency of the motion determined with the L method is equal to 1.6020, 1.6020 and 1.5919 for $k = 0.001$, 0.01 and 0.1, thus indicating that very accurate results can be obtained with $k = 0.01$.

The maximum values of $x, \dot{x}$ and $\ddot{x}$ for $0 \leq t \leq 100$, are 1.8916, 2.5738 and 216.2595, respectively, for $k = 0.001$ and 1.8899, 2.5738 and 216.2770, respectively, for $k = 0.01$, and 1.8833, 2.5425 and 213.2259, respectively, for $k = 0.1$. 

Example 2. This example corresponds to

\[ \ddot{x} + x = \epsilon (1 - x^2) \dot{x}, \]
\[ x(0) = 0.5, \quad \dot{x}(0) = 0, \]
and $\epsilon = 0.5$.

Fig. 1. $y$ (top left), $\dot{y}$ (top right) and $E(t)/E(0)$ (bottom right) as functions of $t$ and phase diagram (bottom left) obtained with $L$ and $k = 0.01$ for Example 1.

Fig. 2. $y$ (top left), $\dot{y}$ (top right) and $E(t)/E(0)$ (bottom right) as functions of $t$ and phase diagram (bottom left) obtained with $L$ and $k = 0.01$ for Example 2.
Fig. 2 shows that, after an initial transient, both the displacement and velocity tend to a periodic behavior whereas the energy oscillates in a periodic fashion. The maximum displacement, velocity and energy, i.e., \( E(t) = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 \), are 2.0025, 2.2270 and 22.4859, respectively, for \( k = 0.001 \), 2.0025, 2.2271 and 22.4879, respectively, for \( k = 0.01 \), and 2.0059, 2.2356 and 22.6879, respectively, for \( k = 0.1 \). The angular frequency of motion is equal to 0.9847, 0.9847 and 0.9846 for \( k = 0.001 \), 0.01 and 1.0, respectively. Therefore, the L method is very accurate even for \( k = 0.1 \) for which there are about 20 time steps per period.

It is known \([21]\) that the van der Pol equation has a limit cycle whose angular frequency is unity. Therefore, the \( L \) method predicts angular frequencies with differ from the exact one by less than 1% even for \( k = 0.1 \). Since the exact period is \( 2\pi \), there are more than sixty time intervals of length equal to 0.1 for \( k = 0.1 \).

Eq. (10) has been previously studied by Mickens and Gumel \([22]\) who employed a non-standard finite difference method and found that their method converges even for \( k = 0.50 \), whereas it results in overflow for \( k = 0.55 \). By way of contrast, these authors also found that the explicit forward Euler method does produce overflow for \( k \geq 0.51 \). On the other hand, the \( L \) method presented here did not result in overflow for \( k = 1 \), although the angular frequency predicted by this time step was 2.2306.

**Example 3.** This example corresponds to

\[
\ddot{x} + x + \epsilon x^3 = 0, \tag{12}
\]

\[
x(0) = x_0, \quad \dot{x}(0) = 0, \tag{13}
\]

and \( \epsilon \) is zero or positive real number. The total energy of the system, i.e., \( E(t) = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{4} x^4 = E(0) \) is conserved. In order to solve this problem for various \( x_0 \), we set \( x = x_0 \epsilon \) and obtain

\[
\ddot{z} + z + \epsilon z^3 = 0, \tag{14}
\]

\[
z(0) = 1, \quad \dot{z}(0) = 0, \tag{15}
\]

where \( x = x_0 \epsilon \).

This equation has a limit cycle solution which has been determined by Lim and Wu \([13]\) by means of an iterative harmonic balance method. In Table 1, we show a comparison between the results obtained by means of the harmonic balance method \([6]\), Mickens’ iterative harmonic balance procedure \([23]\), the two iterative harmonic balance techniques proposed by Lim and Wu \([13]\), and the results of \( L \) with \( k = 0.0001 \).

The angular frequencies predicted by \( L \) for \( x(0) = 1, \dot{x}(0) = 0 \) and \( \epsilon = 0.1 \) are 1.3185, 1.3175, 1.3095 and 1.0112 for \( k = 0.001, 0.01, 0.1 \) and 1, respectively. The maximum and minimum values of \( \omega \) for \( 0 \leq t \leq 100 \) are equal to 1 and 0.9999, respectively, for \( k = 0.001 \) and 0.01, and 1 and 0.9632, respectively, for \( k = 0.1 \). Therefore, \( L \) preserves the energy very accurately for \( k \leq 0.01 \), as indicated in Fig. 3.

**Example 4.** This example corresponds to

\[
\ddot{x} + \frac{x^3}{1 + x^2} = 0, \tag{16}
\]

\[
x(0) = 1, \quad \dot{x}(0) = 0, \tag{17}
\]

which is a conservative system, i.e., \( E(t) = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 - \frac{1}{2} \ln(1 + x^2) = E(0) \).

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( \omega ) ([6])</th>
<th>( \omega ) ([23])</th>
<th>( \omega ) ([13])</th>
<th>( \omega ) ([13])</th>
<th>( \omega ) (L)</th>
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</tr>
<tr>
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<td>1.24750</td>
<td>1.26491</td>
<td>1.26112</td>
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</tr>
<tr>
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<td>1.29297</td>
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<td>1.58114</td>
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</tr>
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</tr>
</tbody>
</table>
For this example, Mickens [24] proposed a non-standard finite difference method here referred to as M that preserves the invariance of the Hamiltonian under reflections in $x$ and its conservation characteristics. This method can be written as
\begin{equation}
x_{n+2} = \frac{2x_{n+1}}{1 + (\phi^2/2)x_{n+1}^2/(1 + x_{n+1}^2)} - x_n,
\end{equation}
where $x_0 = x(0)$ and $x_1$ is determined from Taylor’s series expansion as
\begin{equation}
x_1 = x_0 - \frac{x_0^3}{1 + x_0^2} \frac{k^2}{2} + O(k^4),
\end{equation}
where $\phi = \frac{n}{2p(1/\sqrt{2n/\pi})}$ and $F$ denotes the complete elliptic integral of the first kind. This value of $\phi$ is related to the frequency of the nonlinear oscillator $\ddot{z} + z^3 = 0$ which can be solved exactly.

The angular frequencies predicted by L are 0.6371, 0.6366, 0.6363 and 0.4670 for $k = 0.001$, 0.01, 0.1 and 1, respectively, whereas those of M are 0.6371, 0.6366, 0.6351 and 0.5087, respectively, for the same time steps. These results indicate that L and M yield the same periods for $k = 0.01$ and 0.01, and that the accuracy of the former is higher than that of the latter for $k = 0.1$; however, the accuracy of L deteriorates faster than that of M as $k$ is increased above $k = 0.1$.

The results presented in Fig. 4 indicate that L preserves very well the energy for $k = 0.01$; similar results were found for $k = 0.001$. For $k = 0.1$, however, the energy was found to decrease and reached a value of about $0.002$. In fact, the maximum and minimum values of $E(t)/E(0)$ were found to be 1 and 1, respectively, for $k = 0.001$ and 0.01, and 1 and 0.9955, respectively, for $k = 0.1$.

Example 5. This example corresponds to
\begin{align}
\ddot{x} + x &= \epsilon(1 - x^2) \text{sign}(\dot{x}), \\
x(0) &= 0, \quad \dot{x}(0) = x,
\end{align}
and $\epsilon$ a positive real number. This equation has been previously analyzed by means of the method of slowly amplitude and phase [25] which shows that all the solutions approach
\begin{equation}
x(t, \epsilon) \approx \sqrt{\frac{8}{3}} \cos(t + \phi),
\end{equation}
where $\phi$ is a constant phase and $\sqrt{8/3} \approx 1.633$. Mickens et al. [25] used a non-standard method here referred to as $M$ for the solution of Eq. (20). This method can be written, after correcting some typographical mistakes in the original publication, as

$$x_{n+1} = \psi x_n + \phi y_n,$$

$$y_{n+1} = \psi y_n + \phi (-x_{n+1} + \epsilon (1 - x_{n+1}^2) \text{sign}(y_n)),$$

where $\psi = \cos(k)$ and $\phi = \sin(k)$.

Fig. 4. $y$ (top left), $\dot{y}$ (top right) and $E_{\infty}/E_0$ (bottom right) as functions of $t$ and phase diagram (bottom left) obtained with $L$ and $k = 0.01$ for Example 4.

where $\phi$ is a constant phase and $\sqrt{8/3} \approx 1.633$. Mickens et al. [25] used a non-standard method here referred to as $M$ for the solution of Eq. (20). This method can be written, after correcting some typographical mistakes in the original publication, as

$$x_{n+1} = \psi x_n + \phi y_n,$$

$$y_{n+1} = \psi y_n + \phi (-x_{n+1} + \epsilon (1 - x_{n+1}^2) \text{sign}(y_n)),$$

where $\psi = \cos(k)$ and $\phi = \sin(k)$.

Fig. 5. $y$ (top left), $\dot{y}$ (top right) and $E_{\infty}/E_0$ (bottom right) as functions of $t$ and phase diagram (bottom left) obtained with $L$ and $k = 0.01$ for Example 5 with $\epsilon = 0.1$ and $\epsilon = 0.5$. (The solid and dashed lines denote the results obtained with $L$ and $M$, respectively.)
The use of L and M with $\alpha = 0.1$ and $\epsilon = 0.5$ yields the results illustrated in Fig. 5 that indicate that the limit cycle is reached from the inside and, after an initial transient, the displacement velocity and energy, i.e., $E(t) = \frac{1}{2}x^2 + \frac{1}{4}v^2$, become periodic functions of time. The angular frequency predicted by L for $\alpha = 0.1$ and $\epsilon = 0.5$ is equal to 0.9283, 0.9328 and 0.9832 for $k = 0.001$, 0.01 and 0.1, respectively, whereas those predicted by M are 0.9291, 0.9380 and 1.0016 for the same values of the time step.

The amplitude of the displacement predicted by L is 1.7321, 1.7322 and 1.7455 for $k = 0.001$, 0.01 and 0.1, respectively, whereas that predicted by M is 1.7297, 1.7091 and 1.5229 for the same time steps. On the other hand, the maximum values of $\frac{\delta E_0}{\delta t}$ predicted by L are 366.6762, 366.7594 and 373.6425 for $k = 0.001$, 0.01 and 0.1, respectively, whereas those predicted by M are 365.5697, 355.9341 and 277.0722 for the same time steps. Based on these results, it may be stated that L is more accurate than M even for time steps as large as $k = 0.1$, for which M predicts a smaller limit cycle than L. Moreover, the results presented in Fig. 5 also indicate that M predicts a slightly higher frequency than L.

For $\alpha = 2$ and $\epsilon = 0.5$, the results exhibited in Fig. 6 indicate that the dynamics settles into a limit cycle (which should be identical to the one shown in Fig. 5) from the outside. However, due to the linearization employed in obtaining L, there are some discrepancies in the angular frequency and amplitude of the limit cycle corresponding to the conditions of Figs. 5 and 6. In fact, for $\alpha = 2$ and $\epsilon = 0.5$, the angular frequency predicted by L is equal to 0.9288, 0.9301 and 0.9886 for $k = 0.001$, 0.01 and 0.1, respectively, whereas that predicted by M is equal to 0.9290, 0.9352 and 1.0163 for the same time steps. The slight differences between the frequencies of L and M can be observed in Fig. 6.

The amplitude of the displacement predicted by L is 1.7723, 1.7724 and 1.7877 for $k = 0.001$, 0.01 and 0.1, respectively, whereas that predicted by M is equal to 1.7703, 1.7522 and 1.5840 for the same time steps. This indicates that the accuracy of M deteriorates at a faster pace than that of L as the time step is increased. In fact, M predicts a smaller limit cycle than L. Moreover, the maximum values of the energy predicted by L are 1.1667, 1.1667 and 1.1701 for $k = 0.001$, 0.01 and 0.1, respectively, whereas those predicted by M are equal to 1.7703, 1.7522 and 1.5840 for the same time steps, thus indicating that L is more accurate in the prediction of the energy than M.

The slight discrepancies between the limit cycles shown in Figs. 5 and 6 decrease as the time step is decreased and are associated with the fact that the right hand side of Eq. (20) is not differentiable at $\dot{x} = 0$. In addition, the very steep slope of the phase diagram near the maximum and minimum values of $x$ suggests the use of variable time steps whose values can be adjusted as the solution evolves, i.e., small time steps should be employed where the displacement and/or the velocity exhibit large variations.

Fig. 6. $y$ (top left), $\dot{y}$ (top right) and $\frac{\delta E_0}{\delta t}$ (bottom right) as functions of $t$ and phase diagram (bottom left) for $k = 0.01$ for Example 5 with $\alpha = 2$ and $\epsilon = 0.5$. (The solid and dashed lines denote the results obtained with L and M, respectively.)
4. Conclusions

A piecewise linearization method based on the Taylor series expansion of nonlinear ordinary differential equations with respect to time, the displacement and velocity, has been presented and applied to smooth and non-smooth, conservative and non-conservative, nonlinear oscillators that have limit cycles. The method provides analytical solutions in open intervals that are smooth globally, and results in nonlinear maps for the displacement and velocity in terms of the nonlinearity and its partial derivatives with respect to time, displacement and velocity. It also yields non-standard finite difference equations in a unified manner and allows to vary the step size according to the evolution of the solution.

By means of five examples, it has been shown that the piecewise linearization method presented in this paper is more robust and yields more accurate solutions than the harmonic balance procedure, the method of slowly varying amplitude and phase, and other non-standard finite difference methods. The latter are usually developed in a case-by-case manner according to certain general rules and accounting for the symmetries of the ordinary differential equations that govern the dynamics of single degree-of-freedom, nonlinear oscillators, including the Hamiltonian for conservative problems. By way of contrast, the method presented here is a general one, provides a unified approach to the study of the nonlinear dynamics of single degree-of-freedom systems, is exact for linear ordinary differential equations with constant coefficients and right-hand sides that are linear functions of time, yields smooth solutions everywhere, and can be applied to virtually any nonlinear, second-order differential equation.

It has also been shown that the accuracy of the piecewise linearization method presented here as well as that of other non-standard methods deteriorates as the time step is increased. However, the method can use variable time steps whose values can be adapted to the evolution of the solution.

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