Annular liquid jets in zero gravity

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The equations describing the steady-state behavior of long annular liquid jets and liquid membranes in zero gravity are solved analytically as a function of the pressure difference across the jet or membrane, Weber number, and nozzle exit angle. The ranges of the parameters for which the analytical solutions are valid are determined, and analytical solutions of the mass absorption rate are obtained as a function of the Peclet and Weber numbers, nozzle exit angle, pressure difference, and thickness of the annular liquid jet. It is shown that the convergence length of annular liquid jets and liquid membranes increases as the Weber number, nozzle exit angle, and pressure coefficient are increased. It is also shown that the mass absorption rate increases as the nozzle exit angle, pressure coefficient, and Weber number are increased; however, the mass absorption rate decreases as the Peclet number and annular jet initial thickness-to-radius ratio are increased.

Keywords: zero gravity, annular jets, mass absorption

Introduction

Vertical annular liquid jets can be used as protection systems in inertial confinement laser fusion reactors and as chemical reactors. The dynamics of annular liquid jets in gravitational environments have been studied by the author, who used Lagrangian coordinates, as a function of the Froude and Weber numbers, pressure difference, nozzle exit angle, and initial thickness to radius ratio. Reference 1 reports analytical and numerical solutions for annular liquid jets in gravitational environments; however, the analytical solutions are valid only for long annular liquid jets.

Ramos determined the range of parameters for which the analytical solutions reported in Reference 1 are valid and used perturbation methods to determine the asymptotic behavior of the liquid jet convergence time, that is, the time at which the liquid converges onto the symmetry axis. Ramos also presented comparisons between experimental data and analytical and numerical solutions as a function of the Froude number, nozzle exit width-to-radius ratio, and pressure difference across the liquid jet.

In gravitational environments, annular liquid jets may exhibit an unsteady behavior characterized by temporal oscillations in the volume enclosed by the annular jet at high Froude numbers. In addition, gravity accelerates the liquid, and the annular jet may become turbulent. As the magnitude of the gravitational acceleration is decreased, the length and volume enclosed by the annular jet and the liquid velocity decrease, and the flow may remain laminar.

Annular liquid jets in zero gravity may experience hydrodynamic instabilities when a gas is injected into the volume enclosed by the jet. These instabilities are characterized by large-amplitude axisymmetric oscillations that pinch off the annular jet and lead to the formation of spherical shells.

Spherical shells filled with deuterium and tritium can be used as targets in inertial confinement laser fusion reactors, where they are compressed to fusion conditions by a high-intensity burst of laser energy. Metallic spherical shells can be sintered together to form strong, light materials that have applications in spacecraft structures. Annular liquid jets can also be used as chemical reactors for the reduction of zirconium from zirconium tetrachloride and sodium, control of toxic wastes, scrubbing of radioactive and nonradioactive particulates and soluble materials, etc.

In this study we consider annular liquid jets and liquid membranes in zero gravity, use Eulerian coordinates, and obtain analytical solutions for the annular jet mean radius, thickness, and velocity as a function of the pressure difference across the annular jet, the Weber number, and the nozzle exit angle. In addition, the concentration of the gases absorbed by the liquid jet and the mass absorption rate are determined analytically for underpressurized, nonpressurized, and overpressurized annular jets, and the ranges of the parameters for which the analytical solutions are valid, that is, for long jets, are examined.

The starting point of our study is the Navier-Stokes equations that are applied to the gas enclosed by the liquid jet, the annular jet itself, and the gases surround-
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These equations can be integrated from the liquid jet symmetry axis to the inner surface of the jet, across the annular jet, and from the outer surface of the jet to infinity. The resulting equations and the use of interface conditions at the inner and outer surfaces of the annular jet can be substantially simplified if the gases enclosed by and surrounding the liquid jet are assumed to be isothermal, inviscid, and incompressible and if the liquid jet is assumed to be isothermal, incompressible, inviscid, and thin. If these conditions are satisfied, one obtains a system of integrodifferential equations that govern the time-dependent dynamics of annular jets. The integrodifferential character of the equations is due to the fact that the pressure of the gases enclosed by the liquid jet is a function of the mass and volume occupied by those gases.\(^3\)

The assumption of inviscid fluids can be justified. The studies of Ramos and Pitchumani\(^4\) indicate that the gases surrounding an annular liquid jet do not affect the jet dynamics except at low Reynolds numbers and for positive angles at the nozzle exit.

In the next section the equations governing the time-dependent dynamics of annular liquid jets and of liquid membranes are presented. In the third section, analytical solutions are obtained for long annular jets and membranes, and the domains of validity of the analytical solutions are determined as a function of the Weber number, pressure coefficient, and nozzle exit angle. These analytical solutions are used in the fourth section to determine the concentration of the gases absorbed by the liquid jet and the mass absorption rate as a function of the aforementioned parameters, annular jet thickness and Peclet number.

Formulation

The equations governing the fluid dynamics of inviscid annular liquid jets in zero gravity can be written in dimensional form as\(^3\) (Figure 1)

\[
\frac{\partial m^*}{\partial t^*} + \frac{\partial}{\partial z^*} \left( m^* u^* \right) = 0
\]

\[
\frac{\partial}{\partial t^*} \left( m^* u^* u^* \right) + \frac{\partial}{\partial z^*} \left( m^* u^* u^* \right) = 2\sigma \frac{\partial J^*}{\partial z^*} + R^* \frac{\partial R^*}{\partial z^*} (\overline{P} - \overline{P})
\]

\[
\frac{\partial}{\partial t^*} \left( m^* v^* \right) + \frac{\partial}{\partial z^*} \left( m^* u^* v^* \right) = R^*(\overline{P} - \overline{P}) - 2\sigma \frac{\partial^2 J^*}{\partial z^*^2} \left( \frac{\partial R^*}{\partial z^*} \right)
\]

\[
J^* = R^* \sqrt{1 + \left( \frac{\partial R^*}{\partial z^*} \right)^2}^{1/2}
\]

where

\(t^*\) is time; \(z^*\) is the axial coordinate; \(m^*\) is the mass per unit length per unit radian; \(u^*\) and \(v^*\) are the axial and radial velocity components of the annular jet, respectively; \(R^*\) is the mean radius of the annular liquid jet; \(\sigma\) is the surface tension; \(\overline{P}\) and \(\overline{P}\) denote the pressures of the gases enclosed by and surrounding the annular liquid jet, respectively; and the asterisks denote dimensional quantities.

Equations (1) and equations (2) and (3) denote the continuity and axial and radial linear momentum equations, respectively. Equation (4) is the kinematic condition at the mean radius of the annular liquid jet.

In equations (1)–(4) we have neglected the friction of the gases enclosed by and surrounding the annular jet; this approximation is consistent with the results of Ramos and Pitchumani,\(^4\) who showed that the external boundary on the annular jet does not affect the dynamics of the jet except at low Reynolds numbers and for positive angles at the nozzle exit.

Equations (1)–(4) were derived by integrating the Cauchy equations across the annular jet from its inner radius \(R^*_m\) to its outer radius \(R^*_e\) (Figure 1). Therefore the values of \(u^*\) and \(v^*\) in equations (1)–(4) represent average values across the thickness of the annular jet. Equations (1)–(4) are asymptotic to terms \(O(b^*)\) for liquid curtains, and they are exact for liquid membranes for which \(R^*_e = R^*_m = R^*\).

The axial velocity, \(u^*\), is independent of the radial distance \(r^*\) measured from the symmetry axis if the fluids enclosed by and surrounding the liquid are gases whose dynamic viscosities are smaller than that of the liquid. However, equations (1)–(4) are not strictly valid near the nozzle exit where there is a relaxation of the velocity profile from the stick condition in the nozzle to slip conditions at the annular jet interfaces. These interfaces are material surfaces, and the shear stresses must be continuous across them, whereas the difference in normal stresses is balanced out by surface tension.

Figure 1. Schematic of an annular liquid jet

Equations (1)–(4) are subject to the following initial and boundary conditions:

\begin{align}
\dot{m}(0, z^*) &= m^*_m(z^*), \quad R^*(0, z^*) = R^*_m(z^*), \quad u^*(0, z^*) = u^*_m(z^*), \quad v^*(0, z^*) = v^*_m(z^*) \tag{6} \\
\dot{m}(t^*, 0) &= m^*_m(t^*), \quad R^*(t^*, 0) = R^*_m(t^*), \quad u^*(t^*, 0) = u^*_m(t^*), \quad v^*(t^*, 0) = v^*_m(t^*) \tag{7}
\end{align}

where the subscripts “in” and “0” denote initial conditions and conditions at the nozzle exit, that is, at \( z^* = 0 \), respectively.

Under steady-state conditions, equations (1)–(4) can be written as

\begin{align}
\dot{m}^* u^* &= m^*_m u^*_m \\
\dot{m}^* \frac{du^*}{dz^*} &= 2\alpha \frac{dJ^*}{dz^*} + R^* \frac{dR^*}{dz^*} (P - \bar{P}) \\
\dot{m}^* \frac{dv^*}{dz^*} &= R^* (P - \bar{P}) - 2\alpha \frac{dJ^*}{dz^*} \frac{dR^*}{dz^*} \\
v^* &= u^* \frac{dR^*}{dz^*}
\end{align}

(8) (9) (10) (11)

where \( m^*_m \) and \( u^*_m \) in equation (8) are constants, equation (7) represents the boundary conditions at \( z^* = 0 \), and \( v^*_m \) is constant. Note that in steady state, \( m^*_m, R^*_m, u^*_m, \) and \( v^*_m \) are independent of time (cf. equation (7)).

Equation (11) states that the mean radius of the annular jet is a streamline, and equations (5) and (8)–(11) can be nondimensionalized by introducing

\begin{align}
\frac{u}{u_0^*}, \quad \frac{v}{v_0^*}, \quad \frac{R}{R_0^*}, \quad \frac{z}{z_0^*}
\end{align}

(12)

where variables without asterisks denote dimensionless quantities.

Introducing equations (12) and (13) into equations (8)–(11) and (5), one obtains the following system of dimensionless Eulerian equations:

\begin{align}
\frac{du}{dz} &= \frac{1}{\text{We}} \frac{dJ}{dz} - C_p R \frac{dR}{dz} \\
\frac{dv}{dz} &= \frac{1}{\text{We}} \frac{dJ}{dz} \frac{dR}{dz} + C_p R \\
v &= R \sqrt{1 + \left( \frac{dR}{dz} \right)^2} \\
J &= R \sqrt{1 + \left( \frac{dR}{dz} \right)^2}^{1/2}
\end{align}

(14) (15) (16) (17)

Equations (14)–(16) are subject to the following initial conditions:

\begin{align}
u(0) &= 1, \quad v(0) = \frac{v_0^*}{u_0^*} = \tan \theta_0, \quad R(0) = 1
\end{align}

(18)

where \( \text{We} \) is the Weber number, \( C_p \) is the pressure coefficient, and \( \theta_0 \) denotes the angle that the tangent to the annular jet makes with the \( z \)-axis at the nozzle exit, that is, at \( z = 0 \).

**Long annular liquid jets**

For long annular liquid jets, \( |dR/dz| << 1 \) and \( |d^2R/dz^2| << 1 \), and equations (14) and (15) can be approximated by (cf. equation (17))

\begin{align}
\frac{du}{dz} &= 0 \\
\frac{dv}{dz} &= C_p R - \frac{1}{\text{We}} \\
\frac{d^2R}{dz^2} &= -C_p R = -\frac{1}{\text{We}}
\end{align}

(19) (20) (21)

The solution of equation (19) (cf. equation (18)) is \( u = 1 \), and equation (16) can be substituted into equation (20) to obtain

\begin{align}
\frac{d^2R}{dz^2} &= -C_p R = -\frac{1}{\text{We}}
\end{align}

The solution of equation (21) subject to equation (18) is

\begin{align}
C_p = 0: \quad R &= -\frac{z^2}{2\text{We}} + z \tan \theta_0 + 1 \\
C_p > 0: \quad R &= \frac{1}{\text{We} C_p} + \left( 1 - \frac{1}{\text{We} C_p} \right) \cosh (C_p^{1/2} z) + C_p^{-1/2} \tan \theta_0 \sinh (C_p^{1/2} z) \\
C_p < 0: \quad R &= -\frac{1}{\text{We} |C_p|} + \left( 1 + \frac{1}{\text{We} |C_p|} \right) \cos (|C_p|^{1/2} z) + \frac{\tan \theta_0}{|C_p|^{1/2}} \sin (|C_p|^{1/2} z)
\end{align}

(22) (23) (24)
Equations (22)–(24) correspond to nonpressurized (\(P = \bar{P}\)), overpressurized (\(P > \bar{P}\)), and underpressurized (\(P < \bar{P}\)) annular liquid jets, respectively.

In the following subsections the range of parameters for which equations (22)–(24) are valid are determined.

**Nonpressurized annular jets**

Equation (22) indicates that if the liquid jet thickness is zero, that is, for liquid membranes, \(R^* + R^*_0 = R^*\) (cf. Figure 1), the nondimensional convergence length defined as the axial distance \(L\) at which \(R(L) = 0\) is given by the following equation:

\[
\frac{L^2}{2\text{We}} - L\tan\theta_0 - 1 = 0
\]  
(25)

or

\[
L - \text{We}c\left[\tan\theta_0 + \left(\tan^2\theta_0 + \frac{2}{\text{We}}\right)^{1/2}\right]
\]  
(26)

which indicates that the convergence increases as the Weber number and the nozzle exit angle are increased. Note that the root corresponding to the negative sign in equation (25) is physically meaningless, since it implies a negative convergence length. Note also that \(L > 0\) even for negative angles at the nozzle exit. Furthermore, if \(\theta_0 = 0\), equation (26) implies that

\[
L = (2\text{We})^{1/2}
\]  
(27)

and equation (22) is valid provided that \(L > 1\). that is,

\[
\theta_0 = 0: \quad \text{We} >> 1/2
\]  
(28)

\[
\theta_0 \neq 0: \quad \tan\theta_0 >> \frac{1}{2\text{We}} - 1
\]  
(29)

**Overpressurized annular jets**

The convergence length of equation (23), that is, the axial distance \(L\) at which \(R(L) = 0\), is given by the following equation

\[
L = \frac{2}{C_{p}^{1/2}}\frac{\text{arg tanh}\left[\frac{-B \pm (B^2 - A + C)^{1/2}}{A - C}\right]}{\left(A - C \neq 0, \theta_0 \neq 0\right)}
\]  
(30)

where

\[
A = 1 - \frac{1}{\text{We}C_{p}}, \quad B = \frac{\tan\theta_0}{C_{p}^{1/2}}, \quad C = \frac{1}{\text{We}C_{p}}
\]  
(31)

The radicand of equation (30) is positive or zero if \(B^2 \geq A - C\)

\[
\left(\tan\theta_0\right)^2 C_{p}^{1/2} + \frac{2}{\text{We}C_{p}} - 1 \geq 0
\]  
(32)

or

\[
\left(\tan\theta_0\right)^2 C_{p}^{1/2} + \frac{2}{\text{We}C_{p}} - 1 \geq 0
\]  
(33)

If \(A < C\) or \(\text{We}C_{p} < 2\), the radicand in equation (30) is greater than \(B^2\), and the convergence length is given by

\[
L = \frac{2}{C_{p}^{1/2}}\text{arg tanh}\left[\frac{B + (B^2 - A + C)^{1/2}}{C - A}\right]
\]  
\(\left(C > A, \theta_0 \neq 0\right)\)  
(34)

Note that if \(A < C\), the negative sign in front of the square root of the radicand in equation (30) is physically meaningless, since it implies that \(L < 0\). Since \(L > 0\), that is, \(0 \leq \text{tan}(C_{p}^{1/2} L/2) \leq 1\), equation (30) implies that

\[
1 + 2B \leq C - A
\]  
(35)

or

\[
\tan\theta_0 \leq \frac{1}{C_{p}^{1/2}} - \frac{1}{\text{We}C_{p}}
\]  
(36)

Equation (36) implies that \(\theta_0 \leq 0\) if \(2 > \text{We}C_{p} \geq 1\). Furthermore, for \(C > A\) and \(\theta_0 \neq 0\), equation (23) is valid, provided that \((L >> 1, \text{equation (34)})\)

\[
1 + 2B >> (C - A) \text{tan}^2\left(\frac{C_{p}^{1/2}}{2}\right)
\]  
(37)

or

\[
1 + \frac{2\tan\theta_0}{C_{p}^{1/2}} \text{tanh}\left(\frac{C_{p}^{1/2}}{2}\right) >> \left(\frac{2}{\text{We}C_{p}} - 1\right) \text{tan}^2\left(\frac{C_{p}^{1/2}}{2}\right)
\]  
(38)

The slope of \(p = \text{tanh}(C_{p}^{1/2} L/2)\) versus \((C - A)\) tends to infinite as \((C - A)\) tends to zero for a constant value of \(B\).

If \(A > C\) or \(\text{We}C_{p} > 2\), the radicand in equation (30) is less than \(B^2\), and since \(L\) must be a real number, \(B^2 \geq (A - C)\). Furthermore, since \(p = \text{tan}(C_{p}^{1/2} L/2)\) is such that \(0 \leq p \leq 1\), the following conditions must be satisfied:

\[
A - C \equiv -1 + \sqrt{R}
\]  
(39)

\[
\pm (B^2 - A + C)^{1/2} \geq B
\]  
(40)

together with

\[
B^2 > (A - C)
\]  
(41)

Equation (39) implies that

\[
\text{tan}\theta_0 \geq \frac{1}{C_{p}^{1/2}} - \frac{1}{\text{We}C_{p}}
\]  
(42)

and equation (40) is satisfied, provided that \(\theta_0 < 0\). The condition \(\theta_0 < 0\) and equation (42) imply that

\[
1 - \frac{1}{\text{We}C_{p}} > \text{tan}\theta_0 \quad \text{or} \quad A > |B| = -B
\]  
(43)

Furthermore, the value of \((A - C) = 1\) corresponds to \(\text{We}C_{p} = \infty\). Values of \((A - C) > 1\) imply that \(\text{We} < 0\) and are physically unacceptable. Therefore equation (41) implies that \(0 \leq (A - C) \leq 1\) and \(B \geq -1\). In addition, the curves \((A - C) = -1 - 2B\) (equation (39)) and \(B = -(A - C)\) (equation (41)) are
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tangent at \((A - C) = -B = 1\), that is, at \(\text{We}_{C_{p}} = \infty\), and the convergence length for \(0 < (A - C) \leq 1\) is given by the following expression:

\[
p = \tanh \left( \frac{C_{p}^{1/2}L}{2} \right) = \frac{-B - (B^{2} - A + C)^{1/2}}{A - C} \quad (44)
\]

Equation (44) indicates that when \((A - C) = -B = 1\), \(p = 1\), and \(L = \infty\); it also indicates that

\[
\frac{\partial p}{\partial (A - C)} \bigg|_{B = -1} \rightarrow \infty \quad \text{as} \quad (A - C) \rightarrow 0 \quad (45)
\]

Equation (44) also shows that for a fixed value of \(B\), \(p\) tends to \(1/B\) as \((A - C) + 0\), as \(B \rightarrow -1\)

\[
\frac{\partial p}{\partial B} \bigg|_{A - C = -1} \rightarrow \infty \quad \text{as} \quad B \rightarrow -1 \quad (46)
\]

Equation (47) coincides with the expression to be developed below for the case \(A = C\). Note, however, that the slope of \(p\) versus \((A - C)\) at constant \(\text{We}_{C_{p}} = 1\) is valid if \(L \gg 1\)

\[
L \rightarrow \frac{2}{C_{p}^{1/2}} \arg \tanh \left( \frac{1}{2|B|} \right) \quad \text{as} \quad (A - C) \rightarrow 0 \quad (47)
\]

Equation (47) implies that the validity of equation (23) is given by \((L >> 1)\)

\[
-1 - 2B \tanh \left( \frac{C_{p}^{1/2}}{2} \right) \gg (A - C) \tanh^{2} \left( \frac{C_{p}^{1/2}}{2} \right) \quad (48)
\]

or

\[
\frac{2}{C_{p}^{1/2}} \tan \left| \theta_{0} \right| \tanh \left( \frac{C_{p}^{1/2}}{2} \right) \gg 1 + \left( 1 - \frac{2}{\text{We}_{C_{p}}} \right) \tanh^{2} \left( \frac{C_{p}^{1/2}}{2} \right) \quad (49)
\]

There are two curves in the parameter space \((p, A - C, B)\) that deserve further attention. These two curves correspond to \(A = C\) and \(B = 0\) and are analyzed in the following paragraphs.

If \(A = C\) or \(\text{We}_{C_{p}} = 2\), the convergence length is given by

\[
p = \tan \left( \frac{C_{p}^{1/2}L}{2} \right) = -\frac{1}{2B} \quad (50)
\]

which implies that \(B < 0\) \((\theta_{0} < 0)\) and \((p \leq 1) B \leq -\frac{1}{2}\) or \(\tan \left| \theta_{0} \right| \geq C_{p}^{1/2}/2\). The value of \(B = -\frac{1}{2}\) results in \(p = 1\) and \(L = \infty\) and corresponds to the intersection of the curve \(C - A = 1 + 2B\) (equations (35) and (39)) with \(C = A = 0\).

Therefore equation (23) with \(A = C\) is valid if \((L >> 1)\)

\[
1 \gg 2 \tan \left| \theta_{0} \right| \tanh \left( \frac{C_{p}^{1/2}}{2} \right) \quad (51)
\]

If \(\theta_{0} = 0\), then \(B = 0\), and the convergence length is given by

\[
L = \frac{2}{C_{p}^{1/2}} \arg \tanh \left( \frac{1}{\text{We}_{C_{p}}} \right) \quad (52)
\]

subject to

\[
0 \leq \text{We}_{C_{p}}/(2 - \text{We}_{C_{p}}) \leq 1 \quad (53)
\]

which implies that

\[
0 \leq \text{We}_{C_{p}} \leq 1 \quad (54)
\]

Therefore equation (23) with \(\theta_{0} = 0\) and \(\text{We}_{C_{p}} < 1\) is valid if \((L >> 1)\)

\[
\text{We}_{C_{p}} >> 2 \tanh^{2} \left( \frac{C_{p}^{1/2}}{2} \right) \left[ 1 + \tanh^{2} \left( \frac{C_{p}^{1/2}}{2} \right) \right] \quad (55)
\]

Note that if \(\theta_{0} = 0\) and \(\text{We}_{C_{p}} = 1\), equation (23) implies that \(R = 1\), that is, a cylindrical annular jet is obtained.

If \(\theta_{0} = 0\) and \(A = C\) \((\text{We}_{C_{p}} = 2)\), equation (23) reduces to

\[
R = \frac{1}{2} + \frac{1}{2} \cosh \left( C_{p}^{1/2} z \right) \quad (56)
\]

and the annular jet never converges. In fact, for \(\theta_{0} = 0\) \((B = 0)\) and \(\text{We}_{C_{p}} \geq 1\) the annular jet never converges.

The results obtained in this subsection are shown in Figure 2, which illustrates the range of the parameters for which equation (23) is valid. Figure 2(a) illustrates the plan view \((B, x)\), whereas Figures 2(b) and 2(c) illustrate the geometry of the \((p, x, B)\) surface cut by \(B = \text{constant}\) and \(x = \text{constant}\) planes.

Figures 2(b) and 2(c) clearly indicate that the value of \(p = 1\) is reached with an infinite slope at \((A - C) = -B = 1\). Figure 2(c) indicates that the slope of \(p\) versus \(B\) along \((A - C) = \text{constant}\) decreases from infinite as \(B = -1\) as \(B\) is increased, whereas Figure 2(b) indicates that the slope of \(p\) versus \(x\) at \((A - C) = \text{constant}\) decreases from infinite at \(x = 0\) as \(x\) is increased. Details of the \(p\) versus \(x\) curves for \(x\) near zero are shown in Figure 2(d), which indicates that these curves have an infinite slope at \(x = 0\).

Figures 2(b) and 2(c) also indicate that \(p\) decreases along \(x = \text{constant}\) as \(B\) is decreased, and along \(B = -1\) as \(x\) is increased from \(x = -1\). Although not physically possible, Figures 2(a), 2(b), and 2(c) show that for \((A - C) > 1\) and \(B < -1\) as \(x\) is increased from \(x = -1\) the surface \((p, x, B)\) exhibits a fold catastrophe. This fold catastrophe is shown by means of dotted lines, but it is physically meaningless since it implies that \(We < 0\).
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Underpressurized annular jets

The convergence length of equation (24) is given by the following equation:

\[ L = \frac{2}{|C_p|^{1/2}} \arctan \left[ \frac{B + (B^2 + C + A)^{1/2}}{A + C} \right] \]  (57)

where

\[ A = 1 + \frac{1}{\text{We} |C_p|}, \quad B = \frac{\tan \theta_0}{|C_p|^{1/2}}, \quad C = \frac{1}{\text{We} |C_p|} \]  (58)

The radicand in equation (57) is always greater than \( B^2 \); therefore the only sign that is physically acceptable is the positive one (the negative sign would imply that \( L < 0 \)), that is,

\[ L = \frac{2}{|C_p|^{1/2}} \arctan \left[ \frac{B + (B^2 + C + A)^{1/2}}{A + C} \right] \]  (59)

Therefore equation (24) is valid, provided that \( L >> 1 \), that is,

\[
L \approx \frac{2 \tan \theta_0}{|C_p|^{1/2}} \cdot \frac{|C_p|^{1/2}}{2} \cdot \left( 1 + \frac{2}{\text{We} |C_p|} \right) \tan^2 \frac{|C_p|^{1/2}}{2} \]  (60)

Gas absorption by long annular liquid jets

The analytical solutions derived in the previous section are valid for long, steady, annular liquid jets and liquid membranes. For annular jets \( m^* = \rho^* R^* b^* \) is the mass per unit length per unit radian, where \( b^* \) is the liquid jet thickness (Figure 1). Furthermore, the solutions derived in the previous section are based on the assumption that the pressure difference across the annular liquid jet is constant. In practice, some of the gases enclosed by the jet are absorbed by the liquid, and \( \bar{P} \) decreases with time if mass is not injected into the volume enclosed by the annular jet.

In this section the gas absorption by annular liquid jets in zero gravity is determined by assuming that \( \bar{P} \) is constant, that is, a gas flow rate is used to keep the pressure enclosed by the annular jet constant. We will also assume that the Peclet number \( \text{Pe} = u^*_0 R^*_0 / D \), where \( D \) is the mass diffusivity of the gases in the liquid, is large, and we will use variables with asterisks to denote dimensional quantities. Furthermore, it will be assumed that volumetric displacement effects due to gas absorption by the liquid can be neglected and that the liquid can be considered to be incompressible.

At high Peclet numbers the following equations govern the gas concentration in the liquid jet:

Figure 2. Schematic of the range of the parameters for which the analytical solution corresponding to long overpressurized jets is valid...
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\[
\frac{\partial u^*}{\partial z^*} + \frac{1}{r^* \partial r^*} (r^* v^*) = 0 \quad (61)
\]

\[
u^* \frac{\partial c^*}{\partial z^*} + v^* \frac{\partial c^*}{\partial r^*} = \frac{D}{r^*} \left( r^* \frac{\partial c^*}{\partial r^*} \right) \quad (62)
\]

where \(c^*\) is the gas concentration in the liquid and \(u^*\) and \(v^*\) denote the local axial and radial velocity components, respectively.

The inner \((r^* = R^*_f)\) and outer \((r^* = R^*_o)\) surfaces of the annular jet are streamlines. Therefore one can introduce the streamfunction \(\psi^*\) defined by

\[
\frac{\partial \psi^*}{\partial r^*} = u^* r^* \quad \frac{\partial \psi^*}{\partial z^*} = -r^* v^* \quad (63)
\]

which automatically satisfies the continuity equation, that is, equation (61).

We can also introduce the following variable:

\[
y^* = \int R^*_f^2 u^* \, dz^* \quad (64)
\]

and assume that \(u^*\) is only a function of \(z^*\), that is, the fluids surrounding and enclosed by the annular jet have smaller dynamic viscosities than the liquid, and the stick-slip velocity relaxation near the nozzle exit is neglected.

Substitution of equations (63) and (64) into equation (62) yields

\[
u^* R^*_f^2 \frac{\partial c^*}{\partial y^*} = \frac{D}{r^*} \frac{\partial}{\partial \psi^*} \left( r^* \frac{\partial c^*}{\partial \psi^*} \right) \quad (65)
\]

For thin jets, that is, \(b^* \ll R^*\), the value of \(r^*\) is approximately equal to \(R^*\), and equation (65) can be approximated by

\[
\frac{\partial c^*}{\partial y^*} = D \frac{\partial^2 c^*}{\partial \psi^* \partial \psi^*} \quad (66)
\]

Equation (66) is subjected to the following initial and boundary conditions:

\[
c(1,y^*) = 0, \quad \psi(1,y^*) = 0 \quad (67)
\]

\[
c(0,y^*) = c^*_f \quad (68)
\]

\[
c(R^*_o b^*,y^*) = c^* \quad (69)
\]

where \(\psi^* = 0\) and \(\psi^* = u^*_o R^*_o b^*\) correspond to the inner and outer surfaces of the annular jet, respectively; \(c^*_f\) is the gas concentration at the jet inner surface; and \(c^*\) is the gas concentration at the jet outer surface, which has been assumed to be equal to the gas concentration at the nozzle exit.

The gas concentration at the annular jet inner surface depends on the interfacial mass transfer resistance. However, for clean interfaces and for small mass transfer rates an equilibrium approximation can be used, and \(c^*_f = H \bar{P},\) where \(H\) is Henry's constant. Since in the analysis presented in this paper a steady-state approximation was used, \(\bar{P}\) is constant, and therefore \(c^*_f\) is constant. Similarly, \(c^*_e\) is constant.

When the fluids surrounding and enclosed by the annular jet consist of several components, the concentration \((c^*_{k})\) of the \(k\)th component at the inner surface of the annular jet is equal to \(H (P)_{L}^{k}\), where \((P)_{k}\) denotes the partial pressure of the \(k\)th component.

By introducing the nondimensional variables

\[
c = \frac{c^* - c^*_f}{c^*_e - c^*_f}, \quad \psi = \frac{\psi^*}{u^*_o R^*_o b^*}, \quad y = \frac{y^*}{u^*_o R^*_o b^*} \quad (70)
\]

into equations (66)–(69), the following system results:

\[
\frac{\partial c}{\partial y} = \frac{1}{Pe} \left( \frac{R^*_o}{b^*} \right)^2 \frac{\partial^2 c}{\partial \psi^2} \quad (71)
\]

\[
c(\psi = 0, y) = 1 \quad (72)
\]

\[
c(\psi, y = 0) = 0 \quad (73)
\]

\[
c(\psi = 1, y) = 0 \quad (74)
\]

The solution of equations (71)–(74) can be written by using superposition and the method of separation of variables as

\[
c = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \exp \left[ -\frac{1}{Pe} \left( \frac{R^*_o}{b^*} \right)^2 \pi^2 y \right] \sin (n \pi \psi) \quad (75)
\]

The mass absorption rate can be written in dimensional and dimensionless variables as

\[
\dot{m}^* = -2 \pi \int_0^{L^*} DR^*_f \frac{\partial c^*}{\partial z^*} (R^*_f, z^*) \, dz^* \quad (76)
\]

\[
m = \frac{1}{Pe} \left( \frac{R^*_o}{b^*} \right)^2 \left\{ y(L) + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left( \frac{R^*_o}{b^*} \right)^2 \left[ 1 - \exp \left[ -\frac{1}{Pe} \left( \frac{R^*_o}{b^*} \right)^2 \pi^2 y(L) \right] \right] \right\} \quad (77)
\]

where

\[
y(L) = \int_0^{L} R^*_f u \, dz \quad (78)
\]

and the dimensionless gas absorption rate \(\dot{m}\) is given by the following expression:

\[
\dot{m} = \dot{m}^*[2 \pi R^*_o b^* u^*_o (c^*_e - c^*_f)] \quad (79)
\]

If the Peclet number is sufficiently large, and if the
thickness of the gas layer in the jet is much smaller than the thickness $b^*$, equations (71)–(73) remain valid, and equation (74) can be approximated for mass absorption only by

$$c(\psi \sim \infty, y) = 0$$  \hspace{1cm} (80)

and the solution of equations (71)–(78) and (80) can be obtained by using the Laplace transform and written as

$$c = \text{erfc} \left\{ \frac{\psi}{\sqrt{2}} \left[ \frac{R_o}{b^*} \left( \frac{\text{Pe}}{y(L)} \right) \right] \right\}$$  \hspace{1cm} (81)

The nondimensional mass absorption rate corresponding to equations (71)–(73) and (80) is

$$\dot{m} = 2 \frac{R_o}{b^*} \left[ \frac{y(L)}{\pi \text{Pe}} \right]^{1/2}$$  \hspace{1cm} (82)

which indicates that the mass absorption rate increases as $R_o/b^*$ and $\text{Pe}^{-1}$ are increased.

Equations (80)–(82) are based on the assumption that the thickness of the concentration boundary layer is much smaller than the annular jet thickness. This assumption may be violated at low Peclet numbers or for very thin annular jets.

Equations (77) and (82) indicate that the mass absorption rate is a function of $y(L)$ (cf. equation (78)). The value of $y(L)$ can be determined by using the analytical solutions obtained in the previous section for underpressurized, nonpressurized, and overpressurized long annular liquid jets as indicated in the next three subsections.

Note that for annular liquid jets

$$u^* R^* b^* = u^*_R R^*_0 b^*_0$$  \hspace{1cm} (83)

which can be written in nondimensional form as

$$u R^* b^* = \frac{b^*_0}{R^*_0}$$  \hspace{1cm} (84)

where $b = b^*/R^*_0$.

Note also that

$$y(L) = \int_0^L R^*_0 u dz - \int_0^L \left( R - \frac{b}{2} \right)^2 u dz$$  \hspace{1cm} (85)

where $u = 1$ for long annular jets (cf. equation (19)).

**Nonpressurized annular jets**

Substitution of equations (22) and (84) into equation (85) yields

$$y(L) = -2 \frac{b^*_0}{R^*_0} L + C^2 L + \frac{A^2}{2} \left[ L + \frac{1}{2[C^*_p]^{1/2}} \sin \left( 2[C^*_p]^{1/2} L \right) \right]$$

\begin{align*}
+ & \frac{B^2}{2} \left[ L - \frac{1}{2[C^*_p]^{1/2}} \sin \left( 2[C^*_p]^{1/2} L \right) \right] - \frac{2AC}{C^*_p} \sin \left( C^*_p^{1/2} L \right) \\
+ & \frac{2BC}{C^*_p} \left[ \cos \left( C^*_p^{1/2} L \right) - 1 \right] - \frac{1}{2[C^*_p]^{1/2}} \sin \left( 2[C^*_p]^{1/2} L \right) - 1
\end{align*}

\begin{align*}
+ & \left( \frac{b^*_0}{R^*_0} \right)^2 \frac{1}{2[C^*_p]^{1/2}} \left[ \frac{2ax + d}{(4ac - d^2)x + a} - \frac{b}{2a^2} \ln \left( \frac{X}{c} \right) \right] + \left[ \frac{2a}{4ac - d^2} + \frac{d^2 - 2ac}{2a^2} \right]
\end{align*}

\begin{align*}
\cdot \frac{1}{(d^2 - 4ac)^{1/2}} \left[ 2ax + d + (d^2 - 4ac)^{1/2} \cdot \frac{d}{(d^2 - 4ac)^{1/2}} \cdot \frac{d}{(d^2 - 4ac)^{1/2}} - \frac{d}{(4ac - d^2)c} \right]
\end{align*}

where

\begin{align*}
A &= 1 + \frac{1}{\text{We}[C^*_p]}, & B &= \frac{\tan \theta_0}{[C^*_p]^{1/2}}, & C &= \frac{1}{\text{We}[C^*_p]} \\
A &= A + C, & d &= -2B, & c &= C - A \\
X &= ax^2 + dx + c, & x &= \tan \left( [C^*_p]^{1/2} L/2 \right)
\end{align*}

\hspace{1cm} (87)

\hspace{1cm} (88)

\hspace{1cm} (89)

\hspace{1cm} (90)
Annular jets in zero gravity: J. I. Ramos

Overpressurized annular jets

Substitution of equations (24) and (84) into equation (85) yields:

\[ y(L) = -2KL + C'L + \frac{A^2}{2} \left[ L + \frac{1}{2CP} \sinh (2CP^2L) \right] \]
\[ + \frac{B^2}{2} \left[ \frac{1}{2CP} \sinh (2CP^2L) - L \right] + \frac{2AC}{CP} \sinh (CP^2L) \]
\[ + \frac{2BC}{CP} \left[ \cosh (CP^2L) - 1 \right] + \frac{AB}{2CP} \left[ \cosh (2CP^2L) - 1 \right] \]
\[ + \frac{1}{2CP} \left( \frac{b^*}{R^*} \right)^2 \left[ \frac{2ax + d}{(4ac - d^2)x} - \frac{x}{a} + \frac{b}{2a^2} \ln \left| X \right| \right] \]
\[ + \left( \frac{2a}{4ac - d^2} + \frac{2ac - d^2}{2a^2} \right) \cdot Q - \frac{d}{(4ac - d^2)c} \]

where

\[ A = 1 - \frac{1}{We CP} \quad B = \tan \theta_0 \quad C = \frac{1}{We CP} \]
\[ a = A - C \quad d = 2B \quad c = A + C = 1 \]
\[ X = ax^2 + dx + c \quad x = \tanh (CP^2L/2) \]

and

\[ Q = -\frac{2}{2ax + d} + \frac{2}{d} \quad \text{if} \quad d^2 = 4ac \]
\[ Q = \frac{1}{(d^2 - 4ac)^{1/2}} \ln \left| \frac{2ax + d - (d^2 - 4ac)^{1/2}}{2ax + d + (d^2 - 4ac)^{1/2}} \right| \quad \text{if} \quad d^2 > 4ac \]
\[ Q = \frac{1}{(4ac - d^2)^{1/2}} \arctan \left( \frac{ax}{2x + dx} (4ac - d^2)^{1/2} \right) \quad \text{if} \quad 4ac > d^2 \]

Presentation of results

Figures 3–5 show the nondimensional convergence length of liquid membranes, that is, \( R_i = R_o = R \), as a function of the nozzle exit angle, Weber number, and pressure coefficient. Figure 3 indicates that the convergence length increases as the Weber number and the nozzle exit angle are increased. For a nozzle exit angle of -30° the convergence length is almost independent of the Weber number for We > 100. Figure 3 also shows that the analytical solutions derived in this paper are in excellent agreement with the numerical solution of equations (14) and (15) for Weber numbers larger than 10. The numerical solution of these equations was obtained by means of an explicit fourth-order accurate Runge-Kutta method.\(^2\) Figure 3 also shows that the analytical solutions overpredict the convergence length for Weber numbers less than ten.

Figure 4 shows that the convergence length increases as \( C_{pn} = C_p We \) is increased. Figure 4 also shows an excellent agreement between the analytical and numerical solutions for Weber numbers larger than 10. For smaller Weber numbers the analytical solutions presented in this paper overpredict the convergence length, since these solutions neglect the radius of curvature in the \( (r^*, z^*) \)-plane.

Figure 5 shows that the convergence length increases as the pressure coefficient and the nozzle exit angle are increased. The differences between the analytical and numerical solutions decrease as the pressure coefficient is increased but are almost independent of \( \theta_o \) for \( \theta_o < 0^o \) and for a fixed value of \( C_{pn} \).
Figures 6–9 also indicate that the mass absorption rate is small at high Péclet numbers. Furthermore, Figure 9 shows that for \( \frac{b_0}{R_0} = 0.0025 \), that is, thin annular jets, the mass absorption rate is substantial at \( \text{Pe} = 10^6 \). This is to be expected, since the semi-infinite medium approximation for the gas concentration in the jet is comparable to the jet thickness for \( \frac{b_0}{R_0} = 0.0025 \) and low Péclet numbers, and for those conditions a semi-infinite medium approximation is not a valid one.
Conclusions

The fluid dynamics of annular liquid jets and liquid membranes in zero gravity has been analyzed as a function of the Weber number, pressure difference across the jet, and nozzle exit angle. Analytical solutions for long annular jets and membranes have been obtained, and the values of the parameters for which these analytical solutions are valid have been determined for underpressurized, nonpressurized, and overpressurized annular jets and membranes.

The analytical solutions have been used to determine the mass absorption rate by the annular liquid jet as a function of the Peclet number, initial thickness-to-radius ratio, Weber number, pressure coefficient, and nozzle exit angle. The mass absorption rate was calculated analytically by considering finite thickness and semi-infinite annular jets, and it increases as the nozzle exit angle, pressure coefficient, and Weber number are increased. The mass absorption rate decreases as the Peclet number and the initial thickness-to-radius ratio are increased.

The mass absorption rate varies as $\text{Pe}^{-1/2}$ in accordance with laminar boundary layer theory. However, the semi-infinite jet approximation used in this paper is not valid at low Peclet numbers when axial diffusion is important and the gas penetration depth is comparable to the annular liquid jet thickness; such an approximation is also not valid for annular jets whose thickness is comparable to the gas penetration thickness.

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Nomenclature

- $a, c, d$ constants defined in equations (89) and (93)
- $A, B, C$ constants defined in equations (31), (58), (88), and (92)
- $b$ liquid jet thickness
- $c$ gas concentration in the liquid jet
- $C_p$ pressure coefficient
- $C_{mn} = C_o \text{We}$
- $D$ mass diffusivity
- $J$ variable defined in equation (5)
- $L$ convergence length
- $m$ liquid jet mass per radian and per unit length
- $p$ variable defined in equation (50)
- $P$ pressure
- $\text{Pe}$ Peclet number
- $Q$ defined in equations (95)–(97)
- $r$ radial coordinate measured from the symmetry axis
- $R$ mean radius of the liquid jet
- $t$ time
- $u$ axial velocity component
- $v$ radial velocity component
- $\text{We}$ Weber number
- $\alpha, \beta, \gamma, \delta$ variables defined in equations (90) and (94)
- $\theta$ angle between the tangent to the liquid jet mean radius and the $z$-axis
- $\rho$ density
- $\sigma$ surface tension
- $\psi$ stream function
- $e$ outer surface of the liquid jet
- $i$ inner surface of the liquid jet
- $\text{in}$ initial conditions
- $o$ nozzle exit

Greek symbols

- $\theta$ angle between the tangent to the liquid jet mean radius and the $z$-axis

Subscripts

- $\text{g}$ gases enclosed by the liquid jet
- $\text{s}$ gases surrounding the liquid jet
- $\text{r}$ rate
- $* \text{ dimensional quantities}$

References