STRUCTURE THEOREM FOR PRIME RINGS SATISFYING A GENERALIZED
IDENTITY

A. Fernández López, E. García Rus and E. Sánchez Campos

Departamento de Algebra, Geometría y Topología. Facultad de Ciencias.
Universidad de Málaga. MALAGA 29071 (Spain)

Abstract

We prove in this paper a structure theorem for prime rings whose symmetric
ring of quotients has nonzero socle. Then this result is applied to prime rings
satisfying a generalized identity, and to prime rings having an alternate involution.

Introduction.

A well-known Posner's theorem [20] states that a prime ring satisfying a
polynomial identity is a (two-sided) order in a central simple finite-dimensional
associative algebra. Later, Martindale (see [15, Th. 1.13.4] or [21, Th. 7.6.15])
obtained a more general result for prime rings satisfying a generalized identity.
Actuality, if R a is prime ring satisfying a generalized identity, then its symmetric
ring of quotients $Q_s(R)$ is a primitive ring with nonzero socle such that the
skewfield associated to $Q_s(R)$ is finite-dimensional over its centre, and each
generalized identity of R is a generalized identity of $Q_s(R)$ (see [4, 5]).

Prime rings with involution whose central closure has nonzero socle were
considered by Montgomery, who proved in [18] that if R is a prime ring with
involution satisfying: $xx^* = 0$ implies $x^*x = 0$, then either the involution is
positive ($xx^* \neq 0$ for every $0 \neq x \in R$), or it is normal ($xx^* = x^*x$). This result
would be extended in [6] to semiprime rings.

On the other hand, based on ideas from semigroup theory, Fountain and Gould
[10, 11] introduced a notion of order (we call local order) in a ring which need not
have an identity, and gave [11] a Goldie-like characterization of local orders in
simple rings with minimal one-sided ideals, which are the natural infinite-
dimensional extension of simple artinian rings [13, p. 16]. Later, Anh and Marki
[1,2] extended these results to one-sided orders.

We prove in this paper that if \( R \) is a prime ring whose symmetric ring of
quotients \( Q(\mathfrak{c}) \) has nonzero socle, then the set \( \mathfrak{I}(\mathfrak{c}) \) of all elements in \( R \) having
finite (left and right) uniform dimension is an ideal of \( R \) which is a local order in the
socle of \( Q(\mathfrak{c}) \). Then this result is applied to prime rings satisfying a generalized
identity, obtaining as a consequence Posner's theorem already cited.

Another consequence of our main result is a structure theorem for a prime ring
\( R \) with alternate involution \((aH(R, \ast) a\ast = 0 \text{ for some nonzero } a \in R)\). Since such
a ring satisfies a generalized identity with involution in the sense of [6], its
symmetric ring of quotients has nonzero socle. Hence, if every element of \( R \) has
finite (left and right) uniform dimensions then \( R \) is a local order in a simple ring
with alternate involution containing minimal one-sided ideals. We note that this
result can be used to complete the proof of Theorem 1 of [22] (see[8]).

1. Local orders in associative rings.

Let \( R \) be an associative ring (not necessarily with identity). An element \( a \in R \)
is called square cancellable if
\[
\begin{align*}
a^2x &= 0 \Rightarrow ax = 0 \\
x^2 &= 0 \Rightarrow xa = 0.
\end{align*}
\]
for \( x \in R \cup \{1\} \). Let \( S(R) \) denote the set of all square cancellable elements of \( R \).
An element \( a \in R \) is locally invertible if there exists an idempotent \( e \in R \) such
that \( a \) is invertible (in the classical sense) in \( eRe \). The local inverse \( a# \in eRe \) is
precisely the group inverse of \( a \), and it is characterized by the conditions:
\[
(1.1) \quad aa# = a#, \quad a = aa#, \quad a# = a#aa#.
\]
The idempotent \( e \) is unique, \( e = aa# = a#a \), and will be denoted by \( e = P(a) \).
Moreover, \( a \) is locally invertible if and only if \( a \in a2Ra2 \) [9, Th. 6].

Suppose that \( R \) is a subring of a ring \( Q \). By (1.1), if \( a \in R \) is locally
invertible in \( Q \), then \( a \) is square cancellable in \( R \). Conversely,

**Proposition 1.** [10, Prop. 2.6]. Let \( R \) be a ring satisfying dcc on principal
right ideals. Then every square cancellable element \( x \) in \( R \) is locally invertible.

Recall that a subring \( R \) of a ring with identity \( Q \) is a classical order in \( Q \) if
\[
\begin{align*}
(1.2) \quad &\text{every nonzero divisor element } x \text{ in } R \text{ (in short } x \in \text{Reg}(R)) \text{ is invertible in } Q, \text{ and} \\
(1.3) \quad &\text{for every } q \in Q, \, q = a^{-1}b = cd^{-1}, \text{ for } b, c \in R, \, a, d \in \text{Reg}(R).
\end{align*}
\]
A subring $R$ of a ring $Q$ (not necessarily unital) will be called a local order in $Q$ if
\begin{align*}
(1.4) & \quad \text{every } x \in S(R) \text{ is locally invertible in } Q, \text{ and} \\
(1.5) & \quad \text{for every } q \in Q \text{ there exists } x \in S(R) \text{ such that } q \in xQx, \text{ and } xRx \text{ is a}
\end{align*}
classical order in the unital ring $xQx = eQe$ ($e = P(x)$).

$R$ is said to be an weak order in $Q$ if
\begin{align*}
(1.6) & \quad \text{given } q \in Q \text{ there exist } b, c, a, d \in S(R) \text{ such that } q = a#b = cd#.
\end{align*}

A weak order $R$ in a ring $Q$ satisfying (1.4) is called an order. Notice that if $Q$ has an identity then every order $R$ in $Q$ is actually a classical order in $Q$ [10, Theorem 3.4].

**Proposition 2.** If $R$ is a local order in a ring $Q$ then $R$ is an order in $Q$.

**Proof.** Let $q \in Q$. By (1.5), $q \in xQx = eQe$ for some $x \in S(R)$, $e = P(x)$, and $xRx$ is a classical order in $eQe$. Hence $q = a#b = cd#$, with $b, c, e \in R$, and $a, d \in S(R)$.

Let $R$ be a ring, $M$ a left $R$-module and $E$ a submodule of $M$. Then $E$ is essential in $M$ if $E \cap N \neq 0$ for all nonzero submodules $N$ of $M$. A nonzero submodule $U$ of $M$ is uniform if all nonzero submodules of $U$ are essential in $U$. It is well known that $M$ contains no infinite direct sum of nonzero submodules if and only if $M$ contains an essential submodule which is a direct sum of $n$ uniform submodules for some natural number $n$. In this case any direct sum of submodules of $M$ has no more than $n$ summands: $n$ is said to be the uniform dimension of $M$, written $u \text{-dim}M$. Dual definitions hold for right $R$-modules.

We write
\begin{align*}
I_l(R) &= \{ x \in R : \text{u-dim}(Rx) < \infty \}, \\
I_r(R) &= \{ x \in R : \text{u-dim}(Rx) < \infty \}
\end{align*}
and
\begin{align*}
I(R) &= I_l(R) \cap I_r(R).
\end{align*}
It is known that $I_l(R)$ and $I_r(R)$ are left and right ideals of $R$ respectively. By [2, Prop. 2], for a prime ring $R$ with nonzero socle,
\begin{align*}
I_l(R) = I_r(R) = \text{Soc}(R).
\end{align*}
In fact, if $a \in \text{Soc}(R)$ then $aR$ (similarly $Ra$) contains an infinite sequence of orthogonal division idempotents.

For a subset $S$ of a ring $R$ we write $\text{lan}(S)$ (respectively, $\text{ran}(S)$) to denote the left (respectively, right) annihilator of $S$, and $\text{ann}(S) = \text{lan}(S) \cap \text{ran}(S)$. If $S = \{ x \}$ consists of a single element, we simply write $\text{lan}(x)$, $\text{ran}(x)$, $\text{ann}(x)$. Put
\begin{align*}
Z_l(R) &= \{ a \in R : \text{lan}(a) \text{ is essential in } R \}.
\end{align*}
If $\mathbb{Z}_1(R) = 0$ then $R$ is left nonsingular. Right nonsingular rings are defined dually. A ring $R$ is nonsingular if it is both left and right nonsingular. It is known that $\mathbb{Z}_1(R)$ and $\mathbb{Z}_2(R)$ are ideals of $R$.

**Proposition 3.** For a prime ring $R$ the following conditions are equivalent:

(i) $R$ is a local order in a simple ring with minimal one-sided ideals,

(ii) $R$ is an order in a simple ring with minimal one-sided ideals,

(iii) $R$ satisfies the following conditions and their duals

(A) acc on left annihilators $\text{lan}(x), \ x \in R$

(B) $I_1(R) = R$,

(iv) $R$ is left nonsingular and $I_1(R) = R = I_1(R)$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Prop. 2; (ii) $\iff$ (iii) and (iii) $\Rightarrow$ (iv) have been proved in [11, Theorem 1.1], and the implication (iv) $\Rightarrow$ (ii) can be get by using the methods of [12].

(ii) $\Rightarrow$ (i). Suppose that $R$ is an order in a simple ring $Q$ with minimal one-sided ideals. By [1, Prop. 10] and its dual, for every $s \in S(R)$ the prime ring $sRs$ is a classical order in the simple artinian ring $sQs = eQe$ ($e = P(s)$). Thus we need only to prove that every $q \in Q$ is contained in $xQx$ for some $x \in S(R)$. By Litoff's theorem [14, p.19], given $q \in Q$ there exists $u = u^2 \in Q$ such that $q \subseteq uQu$ with $u = u_1 + \ldots + u_n$ a sum of orthogonal division idempotents. This reduces the problem to the case of a division idempotent $q$. Now $q = a#b = cd#$ where $b, c$ can be taken (see [10, Lemma 2.1]) such that $aa#b = b$ and $cd#d = c$. By minimality of the left ideal $Qq$, $Qq = Qb$ and similarly $qQ = Qc$. Since $R$ is prime, there is $x \in R$ such that $s = cxb \neq 0$. Hence $s \in cQ = qQ \Rightarrow sQ = qQ$ and $Qs = Qb = Qq$. Then $q \in qQ = sQs$ with $s \in S(R)$.

Given a semiprime ring $R$, consider the set of all left $R$-module homomorphisms $f : RI \rightarrow R$ where $I$ ranges over all essential ideals of $R$. Two such functions are said to be equivalent if they agree on their common domains. Let $[f, I]$ denote the equivalence class of $f$ and let $Q_1 = Q_1(R)$ be the set of all such equivalence classes. This set with the usual operations is a ring with identity called the left Martindale ring of quotients of $R$. The mapping $a \rightarrow R_a$ ($R_a x = xa$) is an embedding of $R$ into $Q_1$, and for every $q \in Q_1$ there exists an essential ideal $I$ of $R$ such that $Iq \subseteq R$. Now the symmetric ring of quotients is defined as the subring of $Q_1$ $Q_2(R) = \{ q \in Q_1(R) : qI + Iq \subseteq R, \text{for some essential ideal } I \text{ of } R \}$. This is the approach followed in [15 and 19], but there is a different approach (see [3, 17]).
An essentially defined double centralizer on \( R \) is a pair \((f, g)\) where \( f \) is a right \( R \)-module homomorphism from an essential ideal \( I \) of \( R \) into \( R \) and \( g \) is a left \( R \)-module homomorphism of \( I \) into \( R \), and they satisfy the balanced condition \( xf(y) = g(x)y \) for all \( x, y \in I \). Two essentially defined double centralizers \((f_1, g_1)\) and \((f_2, g_2)\) are equivalent: \((f_1, g_1) \sim (f_2, g_2)\) if and only if \( f_1, f_2 \) and \( g_1, g_2 \) coincide in their common domains. The set of all equivalence classes with the usual operations is a ring isomorphic to the symmetric ring of quotients \( Q_s(R) \) of \( R \). We have that the mapping \( a \to (L_a, R_a) \) is now an embedding of \( R \) into \( Q_s(R) \). Notice that if \( R \) is simple then \( Q_s(R) \) is precisely the ring of multipliers of \( R \).

**Remark:** Let \( Q \) be a simple ring with minimal one-side ideals. Notice that the socle of \( Q_s(Q) \) is precisely \( Q \). In fact, if we represent \( Q \) as the simple ring \( F \gamma(X) \) of all finite rank continuous linear operators relative to a dual pair of vector spaces \((X, Y)\) over a division ring \( A \), then \( Q_s(Q) \) is the ring \( L \gamma(X) \) of all continuous linear operators (see [7] or [15, Th. 1.15.4]).

**PROPOSITION 4.** Let \( R \) be a prime ring which is a local order in a simple ring \( Q \) with minimal one-sided ideals.

(i) For every \( 0 \neq q \in Q \) and every nonzero ideal \( I \) of \( R \), \( qIq \cap I \neq 0 \).

(ii) Given \( 0 \neq I \) ideal of \( R \). For every \( q \in Q \), there exists \( x \in I \) such that \( q \in xQx \).

(iii) The symmetric ring of quotients \( Q_s(R) \) of \( R \) can be embedded in \( Q_s(Q) \).

(iv) \( I_q(Q_s(R)) \subset I_q(Q_s(Q)) = \text{Soc}(Q_s(Q)) = Q. \)

**Proof** (i). Write \( q = a^#b = cd^# \). Then \( cd \neq 0 \) and \( ab \neq 0 \). Since \( R \) is prime, there exists \( x \in I \) such that \( cdxb \neq 0 \). Hence \( 0 \neq cdxb = cd^#d^2xa^2a^#b = qd^2xa^2q \in qIq \cap I \).

(ii). Let \( I \) be a nonzero ideal of \( R \). Given \( q \in Q \), there exists an idempotent \( e \) in \( Q \) such that \( q \in eQe \) and \( e = e_1 + \ldots + e_n \) sum of orthogonal division idempotents. By (i) we can take \( 0 \neq x_1 \in e_1e_1 \cap I \). Then \( q \in eQe = xQx \) with \( x = x_1 + \ldots + x_n \in I \).

(iii). Let \((f, g) \in \text{Hom}(I_R, R_R) \times \text{Hom}(R_I, R_R)\), \( xf(y) = g(x)y \) for all \( x, y \in I \). By (ii), given \( q \in Q \), \( q = xw = vx \) where \( x \in I \) and \( w, v \in Q \). Define \( \tilde{f}(q) = \tilde{f}(xw) = f(x)w \). To see that \( \tilde{f} \in \text{Hom}(Q_s(Q), Q_s(Q)) \) we need just to verify that \( \tilde{f} \) is well defined. Suppose that \( q = x_1w_1 = x_2w_2 \), \( x_1, x_2 \in I \), \( w_1, w_2 \in Q \). Now by the common right denominator property [10, Theor. 4.3], \( w_i = a_i^# \), \( a_i \in R \), \( s \in S(R) \). Then \( x_1a_1^#s = x_2a_2^#s \Rightarrow x_1a_1s = x_2a_2s \Rightarrow f(x_1)a_1s = f(x_2)a_2s \Rightarrow f(x_1)a_1^#s = f(x_2)a_2^#s \).
Define $\tilde{g}(q) = \tilde{g}(v x) := v g(x)$. Similarly, $\tilde{g} \in \text{Hom}(Q, Q)$. Now we must prove the balanced condition. Let $q_i = x_i w_i = v_i x_i \in Q$. Then
\[
q_1 \tilde{f}(q_2) = q_1 \tilde{f}(x_2 w_2) = q_1 f(x_2)w_2 = v_1 x_1 f(x_2)w_2 = v_1 g(x_1)x_2 w_2 = \tilde{g}(v_1 x_1)x_2 w_2 = \tilde{g}(q_1)q_2.
\]
Thus $Q_s(R)$ is contained in $Q_s(Q)$.

(iv). By (iii) $Q_s(R)$ is contained in $Q_s(Q)$. Suppose now that $q$ is an element of $Q_s(R)$ which is not in $\text{soc}(Q_s(Q)) = Q$. Then, $Q_s(Q)$ contains an infinite sequence $\{e_n\}$ of orthogonal division idempotents. Write $e_n = qu_n$ with $u_n = u_n e_n \in Q$. Now, for each positive integer $n$, $u_n = a_n s_n #$, with $s_n \in S(R)$, $a_n \in R$ and $a_n s_n # s_n = a_n$. Then
\[
0 \neq qa_n = qa_n s_n # s_n = qu_n s_n \in qQ_s(R) \cap e_n Q_s(Q).
\]
Hence $qQ_s(R)$ contains an infinite direct sum of right ideals of $Q_s(R)$ and therefore $q$ does not lie in $I_f(Q_s(R))$.

2. Prime rings whose symmetric ring of quotients has nonzero socle.

In this section we obtain the following structure theorem for prime rings whose symmetric ring of quotients has nonzero socle.

**Theorem 5.** Let $R$ be a prime ring such that $Q_s(R)$ has nonzero socle. Then

(i) $I(R) = I_f(R) = I(R) = R \cap \text{soc}(Q_s(R))$,

(ii) $I(R)$ is a local order in $\text{soc}(Q_s(R))$,

(iii) $R$ is nonsingular.

**Proof.** We will give the proof in successive steps.

(2.1) *For every nonzero idempotent $e \in \text{soc}(Q_s(R))$, and every nonzero ideal $M$ of $R$, $e = z # z = z z #$ for some $z \in M$ which is locally invertible in $\text{soc}(Q_s(R))$.*

Since every idempotent in the socle is a sum of orthogonal division idempotents, we may assume that $e$ is a division idempotent. Since $e \in Q_s(R)$, there exists a nonzero ideal $I$ of $R$ such that $eI + Ie \subseteq R$. Take $0 \neq x = es \in R$, $0 \neq y = te \in R$ for some $s, t \in I$. Let $0 \neq v \in M$. Since $R$ is prime, there exist $a, b \in R$ such that $xavby \neq 0$. Then
\[
0 \neq z = xavby \in M \cap e\text{soc}(Q_s(R))e.
\]
Since $e$ is a division idempotent, 
\[ z \text{Soc}(Q_s(R))z = e \text{Soc}(Q_s(R))e \]
which implies that $z$ is locally invertible in $\text{Soc}(Q_s(R))$, with $e = P(z)$ as required.

(2.2) Let $q_1, \ldots, q_n \in \text{Soc}(Q_s(R)).$ Then there exists $s \in R$ which is locally invertible in $\text{Soc}(Q_s(R))$ such that $sq_i, q_is \in R$ and $q_i = s^#sq_i = q_is^#$ for $i = 1, \ldots, n.$

By Litoff's theorem there exists an idempotent $e \in \text{Soc}(Q_s(R))$ such that $\{q_1, \ldots, q_n\}$ is contained in $e \text{Soc}(Q_s(R))e$. Let $M$ be a nonzero ideal of $R$ with $q_iM + Mq_i \subseteq R$, for $i = 1, \ldots, n$. By (2.1), $e = P(s)$ for some $s \in M$ which is locally invertible in $\text{Soc}(Q_s(R))$. Hence $q_i = q_is = q_i ss^#$ and similarly $q_i = s^#sq_i$. Since $s = es$, $s$ is in $\text{Soc}(Q_s(R))$.

(2.3) $\text{Soc}(Q_s(R)) \cap R$ is contained in $I(R)$.

Let $q \in \text{Soc}(Q_s(R)) \cap R$ and suppose that $\Sigma \rho_i$ is an infinite sum of right ideals of $R$ contained in $qR$. For each $p_i$, take $0 \neq r_i \in \rho_i$ and consider the sum $\Sigma_i \text{Soc}(Q_s(R))$ of right ideals of $Q_s(R)$. Since $I(Q_s(R))$ agrees with $\text{Soc}(Q_s(R))$, this sum is not direct. Then 
\[ 0 \neq r_1 p_1 = \Sigma_{j=2, \ldots, n} r_j p_j \text{ with } p_j \in \text{Soc}(Q_s(R)), \quad j = 1, \ldots, n. \]
By (2.2), there exists $s$ in $S(R)$ such that $p_j s \in R$ and $p_j = p_j ss^#$, $j = 1, \ldots, n$. Hence 
\[ 0 \neq r_1 p_1 s = \Sigma_{j=2, \ldots, n} r_j p_j s \in p_1 \cap \Sigma_{j \neq i_1} \rho_j \]
implies that the sum $\Sigma \rho_i$ is not direct. Thus $q \in I_s(R)$. Similarly it is proved that $q \in I_s(R)$, and hence $\text{Soc}(Q_s(R)) \cap R \subseteq I(R)$.

(2.4) $I_s(R) \subseteq \text{Soc}(Q_s(R))$.

This is a consequence of the following more general result.

(2.5) For a prime ring $R$, $R \cap I(Q_s(R)) = I(R)$.

Let $a \in R$ be such that $a \not\in I_s(Q_s(R))$. Then $aQ_s(R)$ contains an infinite direct sum $\Theta \rho_n$ of nonzero right ideals of $Q_s(R)$. For each positive $n$, take $0 \neq aq_n \in \rho_n$ and a nonzero ideal $I_n$ of $R$ such that $q_n I_n \subseteq R$. Then $0 \neq aq_n I_n \subseteq aR \cap \rho_n$. 
Hence $aR$ contains an infinite direct sum of nonzero right ideals of $R$. The reverse inclusion is not difficult.

Now it follows from (2.3) and (2.4) that $I_1(R) = I_1(R) = I(R) = R \cap \text{Soc}(Q_s(R))$. We also have by (2.2) that $I(R)$ is a weak order in $\text{Soc}(Q_s(R))$. Hence by [10, Prop. 2.10] and Proposition 3, $I(R)$ is a local order in $\text{Soc}(Q_s(R))$. We must finally show that $R$ is nonsingular, but this will be a consequence of the following general result.

(2.6) Let $R$ be a semiprime ring and $I$ an essential ideal of $R$. Then $Z_r(I) = Z_r(R) \cap I$, and hence $R$ is nonsingular if and only if $R$ contains a nonsingular essential ideal.

Return to the proof of the theorem. Since $R$ is prime, $0 \neq I(R)$ is essential, but by (ii) and [11, Theorem 1.1], $I(R)$ is nonsingular. Thus $R$ itself is nonsingular by (2.6).

Remark. P.N. Ahn and L. Márki have considered in [2] a refinement of the definition of order and proved that a ring $R$ is a left order in a primitive ring with nonzero socle if and only if $R$ is prime, left nonsingular and has a uniform left ideal. Now the assertion (ii) of Theorem 5 can be refined by saying that $R$ is an order (in the new sense of [2]) in the primitive subring of $Q(R)$ generated by $R$ and $\text{Soc}(Q_s(R))$.

3. A structure theorem for primes ring satisfying a generalized identity.

In this section we get an extension of Posner theorem for prime rings satisfying a generalized identity as a consequence of the structure theorem for prime rings whose symmetric ring of quotients has nonzero socle.

Recall (see [21, p.282]) that for a prime ring $R$, the extended centroid $C$ of $R$ is a field equals the centre of the left Martindale ring of quotients $Q_l(R)$. Now let $X$ be a countably infinite set (of "formal variables"). Let us denote as usual by $C<X>$ the free associative algebra over $C$ generated by $X$, and by $Q_l(R)*C<X>$ the free product of the $C$-algebras $Q_l(R)$ and $C<X>$ [15, p.81]. The elements of $Q_l(R)*C<X>$ are called generalized identities (with coefficients in $Q_l(R)$), and $R$ is said to satisfy a given generalized polynomial identity $p$ if $\phi(p) = 0$ for all homomorphisms $\phi: Q_l(R)*C<X> \rightarrow Q_l(R)$ of $C$-algebras such that $f(X) \subset R$ and
\( \phi(q) = q \) for all \( q \) in \( Q_1(R) \). The proof of the assertions of the following theorem can be found in [16, Theorem 2], [4, Theorem 1.10] and [5, Theorem 2].

**Theorem 6.** Let \( R \) be a prime ring satisfying a nonzero generalized identity. Then (a) \( Q_s(R) \) is a prime ring with nonzero socle, and for every division idempotent \( e \) in \( Q_s(R) \), \( eQ_s(R)e \) is a finite-dimensional division algebra over its centre, and (b) each generalized identity of \( R \) is a generalized identity of \( Q_s(R) \).

As a consequence of Theorem 5 and 6, we get the following extension of Posner's theorem [20] for prime ring satisfying a generalized identity.

**Theorem 7.** Let \( R \) be a prime ring satisfying a nonzero generalized identity. Then
(i) \( I_1(R) = I_2(R) = I(R) = R \cap \text{Soc}(Q_s(R)) \), with \( \text{Soc}(Q_s(R)) = F(Y) \) the simple ring of all finite rank linear operators on \( X \) which are continuous relative to a dual pair \((X,Y)\) of vector spaces over a division algebra \( \Delta \) which is finite-dimensional over its centre,
(ii) \( I(R) \) is a local order in \( F(Y) \).

**Corollary 8.** (Posner theorem). Let \( R \) be a prime ring satisfying a polynomial identity over its centroid. Then \( R \) is a classical order in a central simple finite-dimensional associative algebra.

**Proof.** By Theorem 7, \( I(R) \) is a local order in \( \text{Soc}(Q_s(R)) = F(Y) \) with \((X,Y)\) being a pair of dual vector spaces over a division algebra \( \Delta \) which is finite-dimensional over its centre. Since, by Theorem 6, \( Q_s(R) \) satisfies any generalized identity satisfied by \( R \), we may assume that the primitive ring \( Q_s(R) \) satisfies a homogeneous multilinear identity. Hence, by Kaplansky theorem [16, Theorem 4], \( X \) is finite-dimensional over \( \Delta \), which implies that \( \text{Soc}(Q_s(R)) = Q_s(R) \) is a central simple finite-dimensional associative algebra, say \( M_n(\Delta) \). By Theorem 7, \( I(R) = R \cap \text{Soc}(Q_s(R)) = R \cap Q_s(R) \) implies that \( R = I(R) \) is a local order in \( M_n(\Delta) \); but by [10, Theorem 3.5] and Prop. 3, \( R \) is actually a classical order in \( M_n(\Delta) \).

4. Prime rings with alternate involution.

Let \( R \) be an associative ring (2 torsion free). An involution \( * : R \to R \) will be called diagonal if \( aH(R,*)a^* = 0 \Rightarrow a = 0 \) (\( a \in R \)), where \( H(R,*) = \{ a \in R : a = a^* \} \). Otherwise we say that \( * \) is an alternate involution. Notice that if \( R \) is a prime ring with nonzero socle, then every diagonal (respectively, alternate)
involution $*: R \rightarrow R$ can be represented as the adjoint relative to a nondegenerate hermitian (respectively, alternate) form. See structure theorem for prime rings with involution containing minimal one-sided ideals [13, p. 17].

Let $R$ be a prime ring with an involution $*$, and consider the set $X$ of formal variables as the disjoint union of two equipotent sets $Y$ and $Y^*$, where $X_i \rightarrow X_i^*$ is a bijection form $Y$ onto $Y^*$. With this regarding of $X$, the elements of $Q_1(R)\ast C<X>$ are called generalized identities with involution, and $R$ is said to satisfy a given such identity $p$ if $\phi(p) = 0$ for all homomorphisms of $C$-algebras $\phi: Q_1(R)\ast C<X> \rightarrow Q_1(R)$ such that $f(X) \subset R$, $\phi(q) = q$ for all $q$ in $Q_1(R)$ and $\phi(X_i^*) = \phi(X_i)^*$ for all $X_i$ in $Y$.

For the proof of the following theorem, the reader is referred to [6].

**Theorem 9.** Let $R$ be a prime ring with involution $*$ satisfying a nonzero generalized identity with involution. Then (a) the involution $*$ of $R$ extends to an involution of the symmetric ring of quotients $Q_s(R)$, (b) $Q_s(R)$ is a primitive ring with nonzero socle, and for every division idempotent $e$ in $Q_s(R)$, $eQ_s(R)e$ is a finite-dimensional division algebra over its centre, and (c) each generalized identity with involution of $R$ is a generalized identity with involution of $Q_s(R)$.

We note that every prime ring $R$ with an alternate involution $*: R \rightarrow R$ satisfies the generalized identity with involution $p(X_1,X_1^*) = a(X_1 + X_1^*)a^*$, for some nonzero $a \in R$. Since $R$ is prime, $aRa^* \neq 0$ for every nonzero $a \in R$. Hence $p(X_1,X_1^*)$ is a nonzero generalized identity with involution.

**Theorem 10.** Let $R$ be a prime ring with an alternate involution $*: R \rightarrow R$. Then $I(R)$ is a local order in a simple ring $F_V(V)$, where $V$ is an alternate self-dual vector space over a field $K$.

**Proof.** As we have pointed above, $R$ satisfies the nonzero generalized polynomial identity with involution $p(X_1,X_1^*) = a(X_1 + X_1^*)a^*$ for a nonzero $a \in R$. By Theorem 9, $Q_s(R)$ has nonzero socle and satisfies this same identity. Hence, by Theorem 5, $I(R)$ is a local order in a simple ring $Q = F_V(V)$ with $(V, <,>)$ being an alternate self-dual vector space over a field $K$. 
REFERENCES


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