Short note

Leading-order equivalence of two formulations for long, annular liquid membranes

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In a recent paper, the author [1] studied both analytically and numerically the steady fluid dynamics of annular liquid jets by means of the models developed by Boussinesq [2] and Ramos [3]. In this short note, the formulation developed by Lee and Wang [4] for annular liquid membranes, i.e., annular liquid jets of zero thickness, in the absence of gravity is generalized to include the gravitational acceleration and compared with the one developed by Ramos [3] for annular liquid membranes and annular liquid jets. Lee and Wang [4] developed a model to study the fluid dynamics of liquid membranes and studied numerically the formation of hollow spherical shells of liquid that result from the capillary instabilities of such membranes in zero gravity, and derived their equations by considering differential elements of the membrane and establishing conservation of mass and conservation of linear momentum along and normal to the membrane in the absence of gravity. By way of contrast, Ramos [3] considered annular liquid jets and the Euler equations for axisymmetric flows, accounted for gravity, employed an integral formulation, and assumed that the axial and radial velocity components of annular liquid jets are only functions of time and the axial coordinate.

The objective of this short note is threefold. First, the model of Lee and Wang [4] is generalized to account for the gravitational acceleration, and the governing equations are based on the axial and radial velocity components rather than on those along and normal to the membrane and the resulting equations are compared with those of the model of Ramos [3]. Second, a long-wave analysis of the equations is carried out for inertia and capillary, annular liquid membranes. It is shown that the models developed by Lee and Wang [4] and Ramos [3] are governed by the same leading-order equations for long, annular liquid membranes.

**Lee and Wang’s formulation** [4]. The equations derived by Lee and Wang [4] for axisymmetric, annular liquid membranes along and normal to the membrane in the absence of gravity, may be easily generalized to account for gravity and written as

\[
\frac{\partial m}{\partial t} + u \frac{\partial m}{\partial z} + m \cos \theta \left( \frac{\partial u}{\partial z} \cos \theta + \frac{\partial v}{\partial z} \sin \theta \right) = 0,
\]

\[
\frac{\partial R}{\partial t} = v - u \tan \theta,
\]
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = g - \frac{R \sin \theta}{m} \left( \Delta p - 2\sigma \left( \frac{\cos \theta}{R} - \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right) \right),
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial z} = \frac{R \cos \theta}{m} \left( \Delta p - 2\sigma \left( \frac{\cos \theta}{R} - \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right) \right),
\]
where \(m\) denotes the annular liquid membrane’s mass per unit length, \(u\) and \(v\) are the liquid’s axial and radial velocity components, respectively, \(\sigma\) is the liquid’s surface tension, \(g\) is the gravitational acceleration, \(R\) is the membrane’s radius, \(t\) is time, \(z\) is the axial coordinate, \(\Delta p\) denotes the difference between the pressure of the gases enclosed by and that of those surrounding the liquid membrane, and \(\tan \theta = \partial R/\partial z\).

Eqs. (1)–(4) denote the continuity equation, the kinematic condition, and the conservation of axial and radial momentum, respectively. For steady, annular liquid membranes, it may be easily shown that Eqs. (1)–(4) reduce to those derived by Boussinesq [2] along and normal to the membrane [5].

If the velocity components, axial coordinate, radius, mass and time are nondimensionalized with respect to \(u_0, R_0, R_0, m_0\) and \(R_0/t_0\), respectively, where the subscript 0 denotes (constant) reference conditions, Eqs. (1)–(4) may be written in nondimensional form as

\[
\frac{\partial \tilde{m}}{\partial \tilde{t}} + u \frac{\partial \tilde{m}}{\partial \tilde{z}} + m \cos \theta \left( \frac{\partial u}{\partial \tilde{z}} \cos \theta + \frac{\partial v}{\partial \tilde{z}} \sin \theta \right) = 0,
\]
\[
\frac{\partial \tilde{R}}{\partial \tilde{t}} = v - u \tan \theta,
\]
\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} + u \frac{\partial \tilde{u}}{\partial \tilde{z}} = \frac{1}{\tilde{F}_r} - \frac{R \sin \theta}{m \tilde{W}_e} \left( C_{pn} - \frac{\cos \theta}{\tilde{R}} + \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right),
\]
\[
\frac{\partial \tilde{v}}{\partial \tilde{t}} + u \frac{\partial \tilde{v}}{\partial \tilde{z}} = \frac{R \cos \theta}{m \tilde{W}_e} \left( C_{pn} - \frac{\cos \theta}{\tilde{R}} + \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right),
\]
where \(\tilde{F}_r = (u_0^2/gR_0)\) and \(\tilde{W}_e = (m_0u_0^2/2\sigma R_0)\) denote the Froude and Weber numbers, respectively, \(C_{pn} = (\Delta p R_0/2\sigma)\), and the same symbols have been used for dimensional and nondimensional quantities for the sake of brevity.

Ramos’s formulation [3]. Here, the formulation presented above is compared with that of Ramos [3] who used an integral approach and Taylor’s series expansions in the derivation of the equations which govern the fluid dynamics of both annular liquid jets and annular liquid membranes.

The equations derived by Ramos [3] for slender and thin, annular liquid jets may be written in nondimensional form as

\[
\frac{\partial \tilde{m}}{\partial \tilde{t}} + u \frac{\partial \tilde{m}}{\partial \tilde{z}} + m \frac{\partial \tilde{u}}{\partial \tilde{z}} = 0,
\]
\[
\frac{\partial \tilde{R}}{\partial \tilde{t}} = v - u \tan \theta,
\]
\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} + u \frac{\partial \tilde{u}}{\partial \tilde{z}} = \frac{1}{\tilde{F}_r} - \frac{R \tan \theta}{m \tilde{W}_e} \left( C_{pn} - \frac{\cos \theta}{\tilde{R}} + \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right),
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial z} = \frac{R}{m \text{We}} \left( C_{pn} - \frac{\cos \theta}{R} + \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right).
\]

Comparison of Eqs. (5)–(8) and Eqs. (9)–(12) shows that only the kinematic condition is identical (cf. Eqs. (6) and (10)) in the models of Lee and Wang [4] and Ramos [3]; while the differences between the other equations are expected to be small for small values of \( \theta \), i.e., for long or slender, thin, annular liquid membranes.

Long-wave analysis. If the radius, axial velocity component, axial coordinate, radial velocity component, time, and mass per unit length of Eqs. (1)–(4) are nondimensionalized with respect to \( R_0, u_0, \lambda, \epsilon u_0, \lambda/u_0, \) and \( m_0 \), respectively, where \( \lambda \) denotes a wave length in the axial direction, \( \epsilon = (R_0/\lambda)^{1/2} \), and the subscript 0 denotes (constant) reference values, Eqs. (1)–(4) may be written as

\[
\frac{\partial m}{\partial t} + u \frac{\partial m}{\partial z} + m \cos \theta \left( \frac{\partial u}{\partial z} \cos \theta + \epsilon^2 \frac{\partial v}{\partial z} \sin \theta \right) = 0,
\]

\[
\frac{\partial R}{\partial t} = v - u \tan \theta,
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = \frac{1}{\epsilon^2 \text{Fr}} - \frac{R \sin \theta}{m^2 \text{We}} \left( C_{pn} - \frac{\cos \theta}{R} + \epsilon^4 \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right),
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial z} = \frac{R \cos \theta}{m^2 \text{We}} \left( C_{pn} - \frac{\cos \theta}{R} + \epsilon^4 \frac{\partial^2 R}{\partial z^2} \cos^3 \theta \right),
\]

where

\[
\cos \theta = \frac{1}{\left( 1 + \epsilon^4 \left( \frac{\partial \theta}{\partial z} \right)^2 \right)^{1/2}}, \quad \sin \theta = \frac{\epsilon^2 \frac{\partial \theta}{\partial z}}{\left( 1 + \epsilon^4 \left( \frac{\partial \theta}{\partial z} \right)^2 \right)^{1/2}},
\]

\( \text{Fr} = (u_0^2/gR_0) \) and \( \text{We} = (m_0 u_0^3/2\sigma R_0) \) denote the Froude and Weber numbers, respectively, \( C_{pn} = (\Delta p R_0/2\sigma) \), and the same symbols have been used for the dimensional and dimensionless variables for the sake of brevity. Note that the long wave analysis presented here corresponds to \( \epsilon \ll 1 \), i.e., it corresponds to slender, annular liquid membranes.

Using the same nondimensional variables as those employed above, it can be easily shown that the equations of Ramos' model [3] may be obtained by replacing \( \cos \theta \) and \( \sin \theta \) by 1 and 0, respectively, in Eq. (13), \( (R \sin \theta/m^2 \text{We}) \) by \( (R \tan \theta/m^2 \text{We}) \) in Eq. (15), and \( (R \cos \theta/m^2 \text{We}) \) by \( (R/m^2 \text{We}) \) in Eq. (16).

Depending on the magnitude of the Weber number, several flow regimes may be identified. The inertia regime corresponds to small surface tension and large Weber numbers. For this regime, one may use a characteristic axial velocity \( u_0 \) equal to a (constant) value at the nozzle exit. The capillary regime is characterized by a surface tension larger than inertia, and \( u_0 \) may be chosen equal to the capillary velocity, i.e., equal to \( u_c = (\sigma R_0/m_0)^{1/2} \), which corresponds to \( \text{We} = \frac{1}{2} \), and \( \text{Fr} = (\sigma/gm_0) \) which is the reciprocal of the Bond number.

In the next two paragraphs, these two flow regimes are analyzed.

Inertia regime: If \( \text{We} = W/\epsilon^4, \text{Fr} = F/\epsilon^2 \) where \( W \) and \( F \) are \( O(1) \), Eqs. (13)–(17) indicate that \( \epsilon^2 \) appears in these equations. Therefore, we look for asymptotic expansions for \( R, m, u \) and \( v \) which are powers of \( \epsilon^2 \), i.e.,

\[
R = R_0 + \epsilon^2 R_2 + O(\epsilon^4), \quad m = m_0 + \epsilon^2 m_2 + O(\epsilon^4),
\]
\[ u = u_0 + \epsilon^2 u_2 + O(\epsilon^4), \quad v = v_0 + \epsilon^2 v_2 + O(\epsilon^4), \]

where \( R_i, m_i, u_i, \) and \( v_i, i = 0, 2, 4, \ldots, \) are functions of \( t \) and \( z. \)

Substitution of Eqs. (18) and (19) into Eqs. (13)–(17) yields a system of equations in powers of \( \epsilon^2. \) Equating the coefficients of the terms that multiply \( \epsilon^0 \) yields

\[
\frac{\partial m_0}{\partial t} + u_0 \frac{\partial m_0}{\partial z} + m_0 \frac{\partial u_0}{\partial z} = 0, \tag{20}
\]

\[
\frac{\partial R_0}{\partial t} = v_0 - u_0 \frac{\partial R_0}{\partial z}, \tag{21}
\]

\[
\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial z} = \frac{1}{Fr}, \tag{22}
\]

\[
\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial z} = \frac{R_0}{m_0 W} \left( C_{pn} - \frac{1}{R_0} \right), \tag{23}
\]

which form a system of hyperbolic equations for \( R_0, m_0, u_0 \) and \( v_0. \)

It may be easily shown that substitution of Eqs. (18) and (19) into Eqs. (9)–(12) yields Eqs. (20) and (21); therefore, the leading-order equations of the models developed by Lee and Wang [4] and Ramos [3] are identical. Differences between the two models appear at higher order in \( \epsilon \) due to the differences in the conservation equations already indicated above. Furthermore, for steady state conditions, Eqs. (20)–(23) have the analytical solutions reported by Ramos [3] for \( |\partial R/\partial z| < 1; \) in particular, the leading-order axial velocity component follows Torricelli’s free-fall law. Consequently, these solutions are not presented here, although it must be stated that these analytical solutions are in remarkably good agreement with available experimental data for a variety of flow conditions [5].

Equating the coefficients of powers in \( \epsilon^n \) with \( n \geq 2 \) allows us to determine other terms in the asymptotic expansions of Eqs. (18) and (19).

**Capillary regime.** If \( We = O(1), C_{pn} = O(1) \), and \( Fr = F/\epsilon^4, \) substitution of Eqs. (18) and (19) into Eqs. (13)–(17) results in a system of partial differential equations in powers of \( \epsilon^2. \) Equating the coefficients of the terms in \( \epsilon^0 \) yields

\[
C_{pn} R_0 = 1, \tag{24}
\]

i.e., the annular liquid membrane is a cylindrical one at leading-order.

The model developed by Ramos [3] is also governed by the same equations at leading-order in \( \epsilon. \) Therefore, it may be concluded that the models developed by Lee and Wang [4] and Ramos [4] yield the same leading-order equations for long annular liquid membranes in both the inertia and the capillary flow regimes, and that the differences between the two models appear at second-order in the perturbation parameter. For short, annular liquid membranes, the formulation proposed by Lee and Wang [4] is expected to be more accurate that that of Ramos [4] on account of the differences in the right-hand sides of the momentum equations (cf. Eqs. (3) and (4)).

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References