A statistical model for fatigue crack growth under random loads including retardation effects

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Abstract

A model for the statistical analysis of crack growth under random loading that includes the loading sequence effect is presented. The model defines and incorporates an equivalent closure stress that is included in the fatigue crack growth law via the effective stress intensity factor. The equivalent closure stress for each loading process is obtained from the probability density function of peaks $p(S)$ in the random loading process, the properties of the material and the specimen geometry. The model was applied to the analysis of crack growth life under random loading on sheets of two different aluminum alloys: 2024-T351 and 2219-T851. The crack-growth lifetimes thus obtained were consistent with experimental data and with the results obtained by using a cycle-by-cycle simulation scheme. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Mechanical systems are usually subject to loads of variable amplitude including random loading. If loads are high enough, an initial crack may develop and eventually lead to failure. In order to analyze the fatigue crack growth life under such conditions, the loading process can be described either by representing a sequence of peaks and troughs or by defining several statistical parameters to characterize the loading process. The statistical parameters of choice

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depend on the type of loading process to which the system is subjected. When the loading history is highly uneven, large peaks may delay crack growth and the sequence of applied cycles become important. Under these circumstances, the order in which loading cycles are applied to the system may strongly influence lifetime.

If the sequence effect is not important, the fatigue life can be estimated by using a simple approach. One such approach, perhaps the simplest, is to consider an equivalent loading cycle.
of amplitude computed from peak or range statistics [1] and to apply one of the well-known crack growth rate laws based on linear elastic fracture mechanics. The expected lifetime can also be estimated by using alternative statistical procedures [2,3]. Whatever statistical approach is used, the crack growth life is obtained by direct integration of the crack growth law, using simple statistical parameters such as the equivalent stress level, the distribution of load ranges, the mean load level, etc.

When the sequence effect becomes important, the loading history must be defined in such a way that the real order of the loading cycles can be reproduced. Under these conditions, cycle-by-cycle simulation of crack growth is the most widely accepted procedure for calculating the crack growth life. Several models for the cycle-by-cycle simulation of the crack growth rate that consider sequence effects have been proposed [4–6]. They usually produce better results than the above-mentioned direct integration schemes but are more time consuming and difficult to apply. In some cases, the increased accuracy in the results, by itself, does not justify their use.

In addition to cycle-by-cycle procedures some statistical models also take sequence effects into account. They rely on a global approach that avoids the need for a cycle-by-cycle procedure. Some of these models allow estimation of several statistical parameters of the process. Such is the case with the model of Ditlevsen and Sobićzyk [7], which provides the probability distribution of crack length, and with that of Arone [8], which estimates the lifetime to a final crack length at a preset reliability level. However, these models have some drawbacks. Thus, the model of Ditlevsen and Sobićzyk has not yet been experimentally verified and its parameters are difficult to calibrate. On the other hand, the Arone model is only applicable to a specific type of load variation and could hardly be extended to a more general case. The approach of Veers and Van Den Avyle [9] is wide in scope, however, it provides an estimate of the expected mean life but no other statistical parameter for fatigue life. The model considers the retardation effect by adopting an average value of the closure stress, which is defined as a function of the closure stresses generated in constant amplitude tests and of the type of loading history involved. The ensuing method has been applied to narrow-band gaussian random loading and it is easy to implement. However, it lacks criteria for defining the opening stress as a function of the characteristics of the loading process. In fact, the equivalent opening stress must be experimentally fitted individually for each material and type of spectrum loading.

This paper presents a new approach for estimating crack growth life under variable amplitude loading that includes the sequence effect. It relies on an equivalent or representative closure stress that is directly derived from the statistics of the loading process, the material properties and the spectrum geometry. This equivalent closure stress is used to determine fatigue crack growth via the effective stress intensity factor. The model was applied to the analysis of the fatigue crack growth under random loading on sheets of two different aluminum alloys. The crack growth lives thus obtained are consistent with experimental data and with the results of cycle-by-cycle simulation.

2. Description of the model

For the purpose of analyzing fatigue crack growth, an overload can be considered a load
that is high enough to exert appreciable effects on crack growth during subsequent loading cycles. In metals, the sequence effect is due to variations in crack closure stress. In random loading processes where the highest peaks result from the same process as the others, overloads are fairly evenly distributed in time. Under these circumstances, the effect of an overload, although changing during subsequent cycles, is retained until another overload of similar or greater magnitude arises; the new overload increases the crack closure stress, introduces a new effect and cancels the effect of the previous one. If the effect of an overload $S_{\text{ol}i}$ (Fig. 1) is assumed to be an increase in closure stress, $S_{\text{cl}}$, which will vary slowly during subsequent cycles, then this effect will be maintained approximately until another overload occurs, $S_{\text{ol}(i+1)}$, giving rise to a plastic zone encompassing that produced by the previous overload, i.e. until the following expression holds:

$$a_{\text{ol}(i+1)} + r_{\text{ol}(i+1)} > a_{\text{oli}} + r_{\text{oli}}$$

where $a_{\text{oli}}$ is the crack length at the time overload $S_{\text{oli}}$ occurs and $r_{\text{oli}}$ is the size of the plastic zone produced by the overload.

For a given loading history or a statistically defined random loading process, the probability density function (pdf) of peaks, $p(S)$, can readily be obtained (Fig. 2). On the assumptions that overloads are evenly distributed in time and that their magnitude is fairly similar, given an overload, $S_{\text{oli}}$, the average number of cycles between overloads, $N_q$, can be statistically estimated as the inverse of the probability of having a stress peak equal to or higher than $S_{\text{oli}}$. Thus, $N_q$ can be obtained from the following expression:

Fig. 1. Schematic representation of the assumed changes in closure stress with loading cycles.
The basis of the proposed method is the estimation of an equivalent value for the crack closure stress \( S_{\text{clq}} \) such that the resulting crack growth rate will coincide with the real mean crack growth rate. This equivalent closure stress is used to assess the fatigue crack growth life. \( S_{\text{clq}} \) is estimated from the overloads, the closure stress they produce, changes in closure stress between successive overloads and the number of cycles during which the effect of overloads is retained.

The procedure used to assess the crack growth lifetime is as follows:

1. By assuming a tentative defined overload value, \( S_{\text{qi}} \), at a time \( t_i \) \( (S_{\text{ol}} = S_{\text{qi}}) \), the expected number of cycles, \( N_{\text{qi}} \), between overloads equal to or higher than the previous is estimated from Eq. (2).

   \[
   N_{\text{qi}} = \frac{1}{\int_{S_{\text{ol}}}^{\infty} p(S) dS} 
   \]  

2. The closure stress, \( D_{\text{cl},i} \), produced immediately after \( S_{\text{qi}} \) must be defined. It is assumed to be the value resulting from constant amplitude loading with the same maxima and minima as the overload, and is estimated by using one of the various analytical expressions for parameter \( U \) as developed by different authors [10] and given here by:

   \[
   U = \frac{\Delta S_{\text{eff}}}{\Delta S} = \frac{S_{\text{max}} - S_{\text{cl},i}}{S_{\text{max}} - S_{\text{min}}} = f(R, S_{\text{max}}, \sigma_y) 
   \]
where $S_{\text{max}} = S_{qi}$ and $S_{\text{min}}$ are the maximum and minimum value for the cycle considered as an overload, $R = S_{\text{min}}/S_{\text{max}}$, and $\sigma_y$ is the yield limit of the material.

(3) During crack growth between overload, $S_{qi}$ and the next one, the crack closure stress $S_{\text{cl}}(a)$ is assumed to vary linearly between $S_{\text{cl}i}$ and a value $S_{\text{cl}x}$, according to the equation (Fig. 3):

$$S_{\text{cl}}(a) = S_{\text{cl}i} + \frac{S_{\text{cl}x} - S_{\text{cl}i}}{\Delta a_x}(a - a_{\text{ol}i})$$  \hspace{1cm} (4)

where $S_{\text{cl}}(a)$ is the closure stress at any crack length $a$ in between $a_{\text{ol}i}$ and $a_{\text{ol}i} + \Delta a_x$, $\Delta a_x$ is the crack length increase between the two overloads, and $S_{\text{cl}x}$ is the closure stress immediately before the new overload arises at crack length $a_{\text{ol}i} + \Delta a_x$.

The actual evolution of $S_{\text{cl}}(a)$ is more complex than assumed by this linear approach. The shape of the variation law depends on various parameters such as the stress level of the overload, the stress peaks and troughs following the overload, the number of consecutive overloads applied and the stress state (plane strain, plane stress or intermediate situation) [11,12]. A nonlinear variation law, closer to a more general evolution of the crack closure stress, also including a representation of its initial reduction immediately after the overload and the retardation effect, could also be assumed for $S_{\text{cl}}(a)$. However, its variation law would depend on a large number of parameters that would result in a highly complicated model and make it very difficult to accurately approximate the closure stress evolution for every case. A linear variation was thus assumed instead as it was easier to implement and not very different from the real variation law under plane strain conditions [11].

Based on the above-established criterion for defining a new overload (Fig. 1), the closure stress immediately before the new overload, $S_{\text{cl}x}$, was assumed to be equal to that produced by a loading cycle $S_x$ generating a plastic zone reaching the overload plastic zone (Fig. 3). Thus, $S_{\text{cl}x}$ and the crack length immediately before the overload, $a_x = a_{\text{ol}i} + \Delta a_x$, were determined by

![Diagram showing closure stress changes between overloads and definition of the parameters used in Eq. (4).]
selecting a value $S_x$ and applying the following equation:

$$a_{oli} + r_{oli} = a_{oli} + \Delta a_x + r_x$$  (5)

where $r_x$ is the radius of the plastic zone produced by $S_x$.

As shown later on, the $S_x$ value used scarcely influences the equivalent closure stress to be calculated, $S_{cl}$. In any case, it would not be too different from that associated to the actual crack lengths ($a_x$) existing immediately before the overloads are produced. In this model, $S_x$ was taken to be the root mean square of the loading peaks, $S_x = S_{rms}$.

(4) The expected number of cycles between two overloads can be estimated by analyzing the number of cycles the crack takes to grow from the point where an overload is produced to that where a new overload cancels its effect. If the crack length when the new overload occurs is known (for instance, $a_{oli} + \Delta a_x = a_{oli} + r_{oli} - r_x$), then the number of cycles $N_{pi}$ over which an overload $S_{qi}$ exerts its action will be given by

$$N_{pi} = \int_{a_{oli}}^{a_{oli} + r_{oli} - r_x} \frac{da}{F(S - S_{cl}(a), a)p(S)} dS$$  (6)

where $S_{cl}(a)$ is the previously defined value, $F(S - S_{cl}(a), a)$ is the crack growth rate equation, and $a_{oli}$, $r_{oli}$ and $r_x$ are defined in Fig. 3. The resulting $N_{pi}$ value will depend on the selected $r_x$ value. As said in Section 3, $S_x$ should not be too different from the value that actually produces the plastic zone size that reaches the overload plastic zone immediately before the next overload. However, as shown below, mutually compensating effects in the model result in negligible changes in $N_{pi}$ and also in $S_{cleq}$ over a wide range of $S_x$ values. The $r_x$ value adopted to obtain $N_{pi}$ in this model was the size of the plastic zone produced by a stress equal to the root mean square of peaks in the loading history ($S_x = S_{rms}$).

Based on the previous description, given an overload stress $S_{qi}$, the numbers of cycles $N_{qi}$ and $N_{pi}$, two different approximations to the expected value of the number of cycles between overloads can be considered: $N_{qi}$ based on statistical considerations; and $N_{pi}$, based on fatigue crack growth simulation. If $N_{pi}$ is smaller than $N_{qi}$, then, after $N_{pi}$ cycles, the crack will continue to grow and produce a plastic zone, encompassing that formed by the previous overload, and no new overload equal or greater than $S_{qi}$ in magnitude will yet have been produced. Thus, the new overload will be a new stress smaller than the previous one. So the assumed overload, $S_{qi}$, will be higher than the actual one. The proposed model considers the stress value $S_{qi}$ that results in an $N_{pi}$ value coinciding with $N_{qi}$ to be the expected overload stress, $S_{ol}$, i.e. the stress value $S_{qi}$ at which the expected number of cycles having a new stress peak equal to or greater than it ($N_{qi}$) to be equal to the number of cycles needed to be involved for the plastic zone to encompass that produced by the previous overload plastic zone. By using an iterative procedure involving repeating steps 1–4 $S_{ol}$ can be readily estimated.

(5) After $S_{ol}$ and $N_{pi}$ have been calculated, the equivalent closure stress, $S_{cleq}$, can be obtained as the constant closure stress that results in the same number of cycles $N_{pi}$ needed for the crack to grow between $a_{oli}$ and $a_{oli} + r_{oli} - r_x$. By using a previously reported model [2], $S_{cleq}$ can be obtained from the following equation:
where $F(S - S_{\text{cleq}}, a)$ is an expression for the crack growth rate.

(6) Based on Eqs. (4)–(7), the magnitude of the equivalent closure stress for a given material and random loading process will depend on the size of the plastic zones produced and the rate of crack growth across such zones. Consequently, for a given loading process, the load $P_{\text{cleq}}$ associated with $S_{\text{cleq}}$ will vary with crack length. This entails repeating the above-described procedure at different crack lengths and fitting the values thus obtained in order to derive an expression of the type $S_{\text{cleq}} = S_{\text{cleq}}(a)$, or, in terms of the load $P$ applied to the specimen, $P_{\text{cleq}} = P_{\text{cleq}}(a)$.

(7) After the equivalent closure stress has been determined, the crack growth lifetime from an initial length $a_0$ to a final length $a_f$ can be calculated from

$$N = \int_{a_0}^{a_f} \frac{da}{\int_{S_{\text{cleq}}(a)}^{\infty} F(S - S_{\text{cleq}}(a), a)p_{\text{max}}(S) \ dS - \int_{S_{\text{cleq}}(a)}^{S_{\text{cleq}}(a)} F(S - S_{\text{cleq}}(a), a)p_{\text{min}}(S) \ dS}$$

where $p_{\text{max}}(S)$ and $p_{\text{min}}(S)$ are the pdf of peaks and troughs, respectively (Fig. 4), which are assumed to be symmetric and to have the mean stress value as the symmetry axis.

The second integral in the denominator is included to correct the first one, which considers the effect of stress ranges with their minimum above the closure stress (Fig. 5). This approach is similar to that proposed by Newman for a cycle-by-cycle procedure [13]. Based on Newman’s proposal, the crack length increment produced by the stress rising from $S_3$ to $S_4$ will be

$$\Delta a = F(S_4 - S_{\text{cleq}}, a) - F(S_3 - S_{\text{cleq}}, a)$$

Fig. 4. Probability density functions for peaks ($p_{\text{max}}$) and troughs ($p_{\text{min}}$), for a Gaussian random process.
This equation was proposed for stress increments with $S_{\min} > S_{\text{cl}}$, but only for cases when the current maximum was higher than the highest prevailing maximum stress since the stress excursion crossed $S_{\text{cl}}$, as does $S_4$ in Fig. 5. A simplified version of this proposal, amenable not only to cycle-by-cycle simulation but also to inclusion in global approaches, involves extension to all ranges with $S_{\min} > S_{\text{cl}}$ irrespective of the particular relationship between $S_{\max}$ and the highest maximum stress since the last $S_{\text{cl}}$ crossing. Using this extension, the crack length increase produced by any number of cycles $n$ with $S_{\min} > S_{\text{cl}}$ can be estimated from

$$\Delta a_n = \sum_{i=1}^{n} [F(S_{\max} i - S_{\text{cl}i}, a) - F(S_{\min} i - S_{\text{cl}i}, a)]$$

Eq. (8) can be considered an equivalent statistical approach for those cases where $S_{\min} > S_{\text{cl}}$.

### 3. Application of the model

The proposed model was validated by applying it to various types of random loading and two aluminum alloys: 2024-T351 and 2219-T851. The results thus obtained were compared with the crack growth lives experimentally obtained using the same loading histories and also with those of cycle-by-cycle simulation.

#### 3.1. Group 1 (G1): aluminum alloy 2024-T351

For aluminum alloy 2024-T351, all analyses were carried out on specimens of the compact tension (CT) type that were 50 mm wide and 12 mm thick. The sides of the specimens were polished in order to facilitate the observation of cracks. Because CT specimens are intended for tensile loading, the zero mean recordings obtained were shifted in order to cancel compressive loads.

To analyze the crack growth behaviour of this alloy, five different stationary, gaussian, random loading processes of different shapes and bandwidths were defined. The loading
processes were defined by their mean load and the power spectral density (psd), \( G(\omega) \), of their variable component. The form of the psd of the variable component in the loading process used is shown in Fig. 6a. By altering the initial (\( \omega_a, \omega_b \)) and final frequencies (\( \omega_c, \omega_d \)) for each band and the \( H/h \) ratio, different random processes with different psds and bandwidths are obtained. The five different shapes were designated A–E.

To analyze the effect of the loading level on the ability of the model to predict the fatigue life, three different loading levels per psd shape were used. Thus, once the frequencies and the \( H/h \) ratio for each type had been defined, three different loadings levels for the variable component of the loading process were obtained by varying \( H \). The loading level was characterized by the root mean square of the variable component. The result was 15 stationary, gaussian random loading processes, each of which was defined by its mean value, the frequencies \( \omega_a-\omega_d \), the \( H/h \) ratios, and the root mean squares of the variable component. The mean load used in this analysis was identical in all cases and equal to 4830 N. Also, for each loading level, the root mean square of the variable component was the same for all five psd shapes. The standard deviation of the loading process adopted for the levels designated P1, P2 and P3 were 1078, 521 and 348 N, respectively.

Table 1 gives the characteristic parameters for each type of spectrum and loading level, as well as the values for the irregularity factor, \( \epsilon \), obtained from each psd tested. Parameters \( P_{1rms}, P_{2rms} \) and \( P_{3rms} \) represent the root mean square of peaks for each case. Parameter \( \epsilon \) relates the frequency of mean crossing with a positive slope and the peak frequency, and can

![Fig. 6. (a) Spectral density function for nominal loads. (b) Sample load history for type A process. (c) Sample load history for type B process. (d) Sample load history for type D process.]
be obtained from the psd function using the following expression:

\[
\epsilon = \sqrt{\frac{m_2^2}{m_0 m_4}} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 G(\omega) \, d\omega}{\int_{-\infty}^{\infty} G(\omega) \, d\omega \int_{-\infty}^{\infty} \omega^4 G(\omega) \, d\omega}}
\] (11)

Near-unity \( \epsilon \) values are typical of narrow-band processes (bandwidth increases as \( \epsilon \) decreases). Also, \( \epsilon = 0.75 \) for white noise. Fig. 6b–d show typical recordings for the processes A, C and E; as can be seen, the loading profile was different in each process.

The probability density function for the peaks in this type of stationary gaussian random process can be obtained directly from the psd using the following expression [14]:

\[
p(S) = \frac{1}{\sigma \sqrt{2\pi}} \sqrt{1 - \epsilon^2} \exp\left[-\left(\frac{S^2}{2\sigma^2}\right)\right] + \frac{S\epsilon}{2\sigma^2} \left[1 + \text{erf}\left(\frac{S}{\sigma \sqrt{2(1 - \epsilon^2)}}\right)\right]
\] (12)

where \( S \) is the applied stress, \( \sigma \) the standard deviation for the process, \( \text{erf}(\cdot) \) the Gauss error function and \( \epsilon \) the above-defined parameter.

The \( f(R, S_{\text{max}}, \sigma_y) \) function used to analytically approximate the closure stress produced by an overload was that reported by Schijve [10] for the aluminum alloy 2024 studied:

\[
U = \frac{\Delta K_{\text{eff}}}{\Delta K} = 0.55 + 0.35R + 0.1R^2
\] (13)

3.2. Group 2 (G2): aluminum alloy 2219-T851

The load histories used in analyses and tests for group G2 were taken from the literature,
and so were the data for the specimen type used, viz. the center crack tension (CCT) type, 6.35 mm thick [15]. The 11 loading histories used were designated M81–M92. Table 2 gives various characteristic parameters for the histories: irregularity factor, $\epsilon$; the mean values of the stress peaks and troughs, $S_{\text{mean}}$, the root mean square of the stress peaks, $(S_{\text{max}})_{\text{rms}}$ and their standard deviation, $S_{\text{max}}$.

The equation used to obtain the closure stress from the overload for the 2219-T851 aluminum alloy was taken from experimental results obtained by Bell and Creager [16]:

$$\frac{K_{\text{cl}}}{K_{\text{max}}} = 0.347 + 0.053(1 + R)^{3.93}$$ (14)

4. Base line data and test details

Apply the model entailed defining the crack growth rate equation and the $S_x$ value to be used to obtain $S_{\text{clex}}$ and $r_x$. The crack growth rate equation used, $F(S-S_{\text{clex}}, \alpha)$, was the law reported by Newman [5]. For group G1, the statistical model was also tested with the Paris equation.

Changes in closure stress, $S_{\text{cl}}(a)$, will depend on the $S_x$ value used to obtain $r_x$ and $S_{\text{clex}}$ from Eqs. (4) and (5). As a result, $S_{\text{clex}}$ will also depend on $S_x$. The effect of $S_x$ on $S_{\text{clex}}$ was analyzed by using different $S_x$ values ranging from $S_{\text{rms}}$ to a stress $S'$ such that the probability $P(S>S')$ was 0.001. The $S_{\text{clex}}$ values obtained with these selected $S_x$ values for group G1 were all in the band $\pm 3\%$ while the predicted crack growth life was in the band $\pm 35\%$. An approach to improved selection of $S_x$ values based on an iterative procedure is currently under development.

4.1. Tests

Tests on group G1 were carried out by using the psd defined as P1-C. Overall, 30 tests were performed by using a different loading history in each; all were generated by numerical simulation [17] from the same psd and were therefore representative of the same random process. Each loading history included 5000 peaks. Tests involved repeatedly applying the same loading history on a servohydraulic push–pull testing machine until a preset crack length was

Table 2
Characteristic parameters for the loading histories in group G2

<table>
<thead>
<tr>
<th>Loading history</th>
<th>M-81</th>
<th>M-82</th>
<th>M-83</th>
<th>M-84</th>
<th>M-85</th>
<th>M-86</th>
<th>M-88</th>
<th>M-89</th>
<th>M-90</th>
<th>M-91</th>
<th>M-92</th>
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<td>$\epsilon$</td>
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<td>0.859</td>
<td>0.963</td>
<td>0.963</td>
<td>0.77</td>
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</tr>
<tr>
<td>$\sigma_{\text{max}}$ (MPa)</td>
<td>28.85</td>
<td>43.27</td>
<td>57.70</td>
<td>26.47</td>
<td>39.70</td>
<td>52.93</td>
<td>26.58</td>
<td>35.44</td>
<td>28.71</td>
<td>43.07</td>
<td>57.43</td>
</tr>
<tr>
<td>$S_{\text{mean}}$ (MPa)</td>
<td>45.74</td>
<td>68.60</td>
<td>91.41</td>
<td>32.04</td>
<td>48.05</td>
<td>64.07</td>
<td>43.22</td>
<td>57.63</td>
<td>40.24</td>
<td>60.37</td>
<td>80.50</td>
</tr>
<tr>
<td>$(S_{\text{max}})_{\text{rms}}$ (MPa)</td>
<td>71.2</td>
<td>106.8</td>
<td>142.4</td>
<td>56.7</td>
<td>85.0</td>
<td>113.5</td>
<td>66.8</td>
<td>89.0</td>
<td>65.5</td>
<td>98.5</td>
<td>131.3</td>
</tr>
</tbody>
</table>
reached [18]. The growth lifetime was taken to be the number of cycles needed for a crack to
grow from an initial length $a_0 = 15$ mm to a final length, $a_f = 25$ mm. For group G2, the
experimental results were taken from the literature [15].

4.2. Simulations

The cycle-by-cycle simulation procedure used for comparison in group G1 was the strip-yield
growth was analyzed from histories representative of the 15 random processes defined at five
different bandwidths and three loading levels per bandwidth. The analysis of crack growth
under loading histories representative of the random process of the P1-C type was carried out
by using the same 30 recordings employed in the tests. On the other hand, the analysis of
growth under representative loads for each of the other 14 types of loading processes was
performed by generating 20 different loads histories; each was used to simulate the growth
process and a mean value was calculated from the recording set for each process.

The parameters of the Paris and Newman equations used by the statistical model and of the
model used by the FASTRAN program were fitted to experimental results previously obtained
by the authors [20,21]. The constraint factor used to account for the state of stress (plane stress
or plain stress) in the Newman model was $a = 1.5$ [21]. The same constraint factor was used to
calculate $r_x$ in the statistical model.

With group G2, Newman’s results of cycle-by-cycle simulations [5] were used for comparison
with the proposed model and with experimental data. In this group, the parameters of the
Newman’s crack growth rate equation used in the statistical model and the constraint factor
used to calculate $r_x$ ($x = 1.6$) were both taken from the literature [5].

![Fig. 7. Comparison of the fatigue lifetimes provided by different procedures: (a) experimentally, under the 30 loading histories from the random process P1-C; (b) by cycle-by-cycle simulation under the same 30 loading histories; and (c) with the statistical model as applied to the same random process.](image)
5. Results

Fig. 7 shows the experimental results and those obtained in the cycle-by-cycle simulation for the 30 loading histories generated from the P1-C random loading process (G1). It also shows the results provided by the proposed model, using two different crack growth rate equations: Newman’s and Paris’. Note that this statistical model led to the same estimated value for all the recordings since it uses statistical values for the whole process. As a result, its graphical form is a horizontal straight line. For easier comparison with these latter results, the figure also shows the experimental and simulated mean values as horizontal dashed lines.

Fig. 8 compares the crack growth lives obtained with the proposed model, $N_S$, for the 15 psd

Fig. 9. Comparison of the experimentally determined number of cycles, $N_E$ [15], with the number of cycles predicted by the proposed model, $N_S$, and the cycle-by-cycle simulation, $N_N$, for group G2 (2219-T851 Al alloy).
(G1) and the mean of the lifetimes obtained in the cycle-by-cycle simulation, \( N_N \). For each psd, the results provided the statistical model based on Newman’s and Paris’ crack growth rate laws are shown.

Fig. 9 compares the experimental results, \( N_E \), with those obtained by Newman [5] using the cycle-by-cycle simulation scheme, \( N_N \), and those of the statistical model based on Newman’s crack growth rate equation, \( N_S \), for the 11 loading histories defined for group G2. The \( N_S/N_E \) ratios, except in two specific cases, were between 0.7 and 1.4.

6. Discussion

6.1. Group 1 (G1)

As can be seen from Fig. 7, the statistical model does not allow one to assess variability, which depends on the specific loading history used and is actually predicted by cycle-by-cycle simulations. For the group of tests shown in the figure, the lifetime predicted by the statistical model was closer to the average of the experimental results than to the average of the cycle-by-cycle simulated lifetimes obtained by using a constraint factor \( a = 1.5 \).

Cycle-by-cycle simulations using different constraint factors below 1.5 and a variable constraint factor for the same random loading process, P1-C, were carried out. The simulations provided longer lifetimes that were closer to the experimental values (e.g. the average of the 30 simulations was 116,508 at \( a = 1.2 \) and 171,141 with a variable constraint factor) [21]. Thus, cycle-by-cycle simulation with a variable constraint factor provides a slightly better approximation to the average of the experimental results than the statistical model based on the Newman crack growth equation. However, scatter in the lifetimes obtained by using a variable constraint factor for the 30 loading histories was much greater than the real scatter, which was very close to that obtained at \( a = 1.5 \). Based on the analysis of a cycle-by-cycle simulation procedure previously tested by the authors [21], a constraint factor \( a = 1.5 \) was adopted for all simulations in this work because it provided the best approximation to the real scatter and a good and conservative enough estimate of the average lifetime.

It is generally accepted that the cycle-by-cycle simulation methods used here provides fairly good estimates of fatigue life under random loading [22]. Because additional experimental data for comparison with the results of the statistical model were unavailable, a comparison of the cycle-by-cycle simulated results with those provided by the proposed statistical model would provide an indication of the accuracy of the statistical model. Thus, Fig. 8 warrants several comments:

- As can be seen, the ratio between the two quantities, \( N_S/N_N \), was always within the range from 0.5 to 2.
- The lifetimes provided by the statistical model differed depending on the particular crack growth rate equation used (Paris or Newman).
- Predictions obtained with the statistical model based on the Paris equation as the crack growth rate law were slightly more conservative than those based on the Newman crack growth rate equation. These latter predictions were closer to the cycle-by-cycle simulation.
results than were the others. Nevertheless, the results for tests P1-C in Fig. 7 reveal that, for this spectrum type and load level, the cycle-by-cycle simulation at $\alpha=1.5$ provided some conservative results and the results of the statistical model based on the Newman crack growth rate equation were even closer, though still conservative.

- The shorter crack growth lifetimes obtained with the statistical model based on the Paris equation relative to the Newman equation may be the result of the crack growth threshold not being considered in the Paris equation. Taking into account that the loading histories used in this analysis contained many cycles with $\Delta K_{\text{eff}}$ values below the fatigue crack growth threshold, the Paris equation produced crack growth under those cycles, whereas the Newman crack growth equation considered no crack propagation.

- With longer crack growth lifetimes, the statistical model tends to give lower $N_{S}/N_{N}$ ratios, whichever crack growth rate equation was used, as the likely result of a large number of cycles (lower stress levels) leading to a more prominent role of the crack growth threshold and its being considered in a different way in each model. The cycle-by-cycle scheme considers the variation of the threshold with the loading cycle parameters. On the other hand, the statistical model either includes no threshold effect in Paris’ law or assumes it to be a constant value throughout crack growth when using the other crack growth rate equation.

- The effect of the variation of crack closure stress with the stress level is seemingly more significant. Newman’s strip-yield model considers the effect of the maximum stress/yield stress ratio on the closure stress/maximum stress ratio, which is not included in Eq. (13), used by the statistical model. The strip-yield model assumes the closure stress/maximum stress ratio to increase with decreasing maximum stress/yield stress ratio. As a result the ratio between the closure stress provided by Newman’s model and by Eq. (13) ($S_{cl}/S_{cl,N}$) increases (and hence the $N_{S}/N_{N}$ ratio decreases) with decreasing stress level. It would be interesting to conduct additional tests involving low loading levels in order to achieve more accurate comparisons.

- Concerning the effect of psd bandwidth or the irregularity factor for the loading histories, the model led to similarly satisfactory results at the five bandwidths considered at each stress level.

With the same probability density function for peaks and troughs, the statistical model will lead to the same lifetime estimate regardless of the peak sequence produced (i.e. it considers overload effects but always assumes overloads to be evenly distributed in time). However, overloads are highly irregularly distributed in some loading histories. On the other hand, the effect of an irregular distribution of overloads on the real lifetime can be predicted from cycle-by-cycle simulations. Thus, the influence of this factor on the error produced by the statistical model can be estimated by comparing the lifetimes obtained by applying the cycle-by-cycle scheme to different loading histories, using the same pdf of peaks but arranged in a different sequence.

In order to estimate the error made by the statistical model for a random loading history of group G1 with a highly irregular distribution of overloads, all 310 loading histories previously used were modified to make the overload sequence more irregular. In each loading history, the cycles with the five highest peaks were removed from the original sequence and consecutively
applied. With this modification, the loading histories had the same pdf of peaks; therefore, the proposed model predicts the same lifetimes with these loading histories as with the originals. However, the rather dissimilar distribution of overloads will alter the actual lifetimes, and also the predictions, of the cycle-by-cycle simulation scheme. A cycle-by-cycle simulation of fatigue crack growth under every modified load history was carried out and the lifetimes obtained compared to the original values. The difference between the average lifetimes obtained with the original and modified loading histories was less than 8% for every psd. Differences were smaller for the lower loading levels. In order to estimate the effect of these overloads on fatigue lifetime, the same simulation scheme was applied to the same loading histories, after removing the five highest peaks from the loading history, however. The average lifetime reduction thus obtained relative to the original loading histories was around 30%. The fact that discarding the five highest peaks results in such a substantial reduction suggest that, for this type of stationary random loading process, where overloads are inherent in the process, their effect is significant but not too marked. Because lifetime variations following rearrangement of load cycles were smaller than 8%, the statistical model should provide reasonably good fatigue lifetime estimates for this type of random loading process, even when overloads are highly irregularly distributed. This inference cannot be extended to every random or variable amplitude loading process, nor to every material, without a prior analysis.

6.2. Group 2 (G2)

As can be seen in Fig. 9, the results for group G2 provided by the statistical model, $N_S$, compared fairly well with the experimental results $N_E$. In two cases that are discussed later on, however, data were not so consistent (the $N_S/N_E$ ratio was close to 0.5).

The predictions of the statistical model, $N_S$, and the results obtained of the Newman model, $N_N$, compared similarly to group G1, with $N_S/N_N$ ratios between 0.9 and 1.4 except in the two above-mentioned cases, where the ratio was close to 0.25.

The two cases where the differences between $N_S$ and $N_E$ were greater than the rest were obtained from the same loading history at two different loading levels (tests M88 and M89), i.e. each was obtained multiplying the same loading sequence by a different coefficient. This loading history contained only 300 cycles that were repeated until failure. It produced an artificial sequence effect every 300 cycles that could not be included in this statistical model as it considered stationary processes with very long loading histories. Taking into account the distribution of peaks in these loading histories, and using the procedure for estimating the average number of cycles between overloads (Sections 2(1)–2(4)), the $N_{q}=N_p$ values obtained for these histories were as follows: 8000 cycles for M88 and 4500 for M89. Consequently, based on the statistical distribution of peaks and loading levels, the statistical model assumes the occurrence of a single overload every 8000 or 4500 cycles, while, in fact, owing to the artificial sequence effect, the actual number of cycles between overloads is more than 10 times smaller. This artificial sequence effect will result in more marked retardation than would longer loading histories obtained from the same random loading process.

The large difference—of opposite sign—also observed between $N_E$ and $N_N$ for the same two loading histories can be explained following Newman [5]. Thus, the constraint factor used ($\alpha=1.6$) simulates a behaviour close to plane stress and results in stronger retardation effects.
This may account for the high $N_N/N_E$ ratios obtained on M88 and M89, where retardation effects dominated.

7. Conclusions

Based on the results, the proposed model allows the duration of the crack growth process to be directly estimated from the statistical parameters for the random loading process and the characteristics of the material concerned.

The model provides estimates that are very similar to the expected mean values, at least for the materials and random process studied here. However, it does not allow one to estimate the lifetime scatter resulting from testing different load histories for the same random process.

Also, for the material and stationary random processes studied in this work (group G1), where overloads arise from the same process, rearranging overloads in a highly irregular distribution results in negligible differences between the lifetimes obtained by cycle-by-cycle simulation and those provided by the proposed statistical model.

The previous conclusion cannot be extended to every type of random or variable amplitude loading process, nor to every material, without a prior analysis. The analysis should thus be extended to other materials and loading processes in order to precisely determine the scope of application of the proposed model.

A careful analysis of the effect of the selected value for $S_x$ for any type of loading and material considered should be concluded. Also, further research is required with a view to establishing a reliable procedure for selecting $S_x$ must be conducted.

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