On the method of modified equations.
IV. Numerical techniques based on the modified equation for the Euler forward difference method

F.R. Villatoro *, J.I. Ramos

Departamento de Lenguajes y Ciencias de la Computación, E.T.S. Ingenieros Industriales, Universidad de Málaga, Plaza El Ejido, s/n, 29013 Málaga, Spain

Abstract

The modified equation method is studied as a technique for the development of new numerical techniques for ordinary differential schemes based on the third modified or (simply) modified equation of the explicit Euler forward method. Both direct-correction and successive-correction techniques based on the modified equation are used to obtain higher-order schemes. The resulting numerical techniques are completely explicit, of order of accuracy as high as desired, and self-starting since the truncation error terms in the modified equation have no derivatives. The methods introduced in this paper are applied to autonomous and non-autonomous, scalar and systems of ordinary differential equations and compared with second- and fourth-order accurate Runge–Kutta schemes. It is shown that, for sufficiently small step sizes, the fourth-order direct-correction and successive-correction methods are as accurate as the fourth-order Runge–Kutta scheme. © 1999 Elsevier Science Inc. All rights reserved.

Keywords: Modified equations; Numerical methods; Finite differences; Deferred corrections; Asymptotic successive-correction methods

1. Introduction

This paper is the fourth of a series dealing with the assessment of the method of modified equations as a means for the analysis of finite difference schemes.
and for the development of new numerical ones, and is devoted to the third modified or (simply) modified equation.

In the third paper of this series [1], the second modified or second equivalent equation was studied as a method for the development of new numerical techniques, the stability of the direct numerical correction of the truncation error terms in the second equivalent equation was studied, and the technique of direct corrections was introduced. In that paper, asymptotic successive-correction techniques based on the second equivalent equation for the development of higher-order, stable numerical methods were studied, and both direct-correction and asymptotic successive-correction techniques were applied to first-order ordinary differential equations and systems of ordinary differential equations. In that paper, it was shown that the accuracy of the resulting numerical methods was comparable to that of Runge–Kutta methods of the same order of accuracy; the region of stability of the direct-correction methods was smaller than that of the Euler forward method, and that of the asymptotic successive-correction methods was only slightly smaller than that of the Euler scheme. Thus, the numerical results indicated the validity of the method of modified equations based on the second equivalent equation for the development of new stable numerical methods.

In this paper, the validity of the modified equation based on the third modified or (simply) modified equation is studied as a method for the development of new numerical techniques. In Section 2, direct-correction techniques are used for the estimation and correction of the truncation error terms in the modified equation. There, it is shown that the resulting numerical methods are completely explicit, and their strong and absolute stability is better than that of the original Euler method.

The asymptotic successive-correction techniques developed in Parts II [2] and III [1] are applied in Section 3 to the correction of the modified equation. The Euler forward method is corrected with linear correction terms which are obtained by asymptotic methods applied to the modified equation. This procedure yields explicit numerical methods and has the advantage that the linear stability conditions for the Euler equation are sufficient for the linear stability of the correction equations; thus, the new technique generates higher-order methods with good linear stability properties.

The direct-correction and asymptotic successive-correction techniques developed in this paper are also applied to systems of ordinary differential equations in Sections 4.1 and 4.2.

In Sections 5 and 6, both the direct-correction and the asymptotic successive-correction techniques developed in this paper are compared with the results of Runge–Kutta schemes for several nonlinear, first-order, ordinary differential equations and second-order systems of ordinary differential equations, respectively. The numerical order, the absolute errors and the computational cost of these techniques are studied in those sections.
Finally, Section 7 is devoted to the presentation of the main conclusions on both the accuracy of the new numerical techniques presented in this paper and the validity of the method of modified equations for initial-value problems in ordinary differential equations.

2. Third modified or (simply) modified equation for ordinary differential equations

As in the first paper of this series [3], we shall consider the simplest first-order, non-autonomous, ordinary differential equation

\[
\frac{du(t)}{dt} = F(u(t), t), \quad u(0) = a,
\]

where \(F\) is a function regular enough so that the problem is well posed, and the simplest numerical finite difference scheme for this equation, i.e., the explicit Euler forward scheme,

\[
U_{n+1} = U_n + \frac{k}{F(t)} F(t), \quad U^0 = a,
\]

where \(t^n = nk\) and \(U^n\) is a numerical approximation to \(u(nk)\).

The third modified or (simply) modified equation for the Euler forward method is the following pseudo-differential equation

\[
U_t = F - \frac{1}{2} k (F_i + FF_U) + \frac{1}{12} k^2 (F_i F_U + 4F_i^2 F_U + 4FF_U^2)
+ 2FF_{Ut} + F^2 F_{UU} - \frac{1}{12} k^3 (F_i F_{UU} + 3F_i^2 F_U + 3FF_{U}^3 + F_i F_{Ut})
+ 3FF_i F_{Ut} + FF_i F_{UU} + 2F^2 F_i F_{UU}) + O(k^4),
\]

where \(F \equiv F(U(t), t)\), the subscripts indicate partial differentiation and \(U(t)\) is a formal analytical continuation of the discrete sequence \(U^n\) so that \(U(nk) = U^n\). Eq. (3) is the expansion of a nonlinear pseudo-differential operator which corresponds to the inversion of the pseudo-differential operator in the left-hand side of the first modified or equivalent equation [3].

The modified equation (3) can be used for the correction of the truncation error terms of the Euler forward scheme, thus allowing for the development of new higher-order methods by using the direct-correction method presented in Part III [1]. In order to obtain a higher-order method, each nonlinear term in the truncation errors must be numerically approximated (with opposite sign) by a finite difference formula. In order to determine the truncation error terms in a correction stage of the direct-correction method, the modified equation for the scheme at the previous stage must be used. Since no derivatives appear in the right-hand side of the modified equation, the stability properties of direct-correction methods for the (third) modified equation are expected to be better than those for the equivalent and second equivalent equations.
In this section, we shall only consider explicit direct-correction schemes. These techniques which are based on the modified equation of the Euler forward method, i.e., Eq. (3), are obtained by using explicit difference formulas for the truncation error terms in the modified equation. Thus, retaining only the $O(k)$ term of the truncation error terms in the right-hand side of Eq. (3) and changing the sign of this term yield the following (differential) correction equation,

$$\frac{dU}{dt} = F + \frac{1}{2}k(F_t + FF_t),$$

which, by using the Euler forward difference scheme, yields the following second-order, explicit difference correction method,

$$U^{n+1} - U^n = kF^n + \frac{1}{2}k^2(F''_t + F^nF'_t),$$

where $F^n = F(U^n, t^n)$, which is self-starting. The characteristic polynomial for the linear stability of Eq. (5) is

$$\xi - 1 = \lambda k + \frac{1}{2}(\lambda k)^2,$$

which shows that this scheme is strongly stable, and its region of absolute stability diagram, which is illustrated in Fig. 1, is greater than that of the Euler method.

In order to obtain a third-order method, the modified equation for the Euler forward scheme, i.e., Eq. (2), truncated to $O(k^2)$ terms cannot be used because the numerical approximation to the first correction equation introduces new

---

Fig. 1. Absolute (linear) stability stars for the explicit direct-correction methods based on the modified equation, i.e., Eq. (2) (dotted line with label 1), Eq. (5) (solid line with label 2), Eq. (8) (dashed line with label 3), Eq. (11) (dot-dashed line with label 4) and Eq. (14) (dotted line with label 5), and the one based on Eq. (15) (solid line with label 6). Only the upper $\lambda$-plane is shown since the diagrams are symmetric with respect to the real $\lambda$-axis.
truncation error terms that must be taken into account as follows. The modified equation corresponding to Eq. (5), i.e.,
\[
\frac{dU}{dt} = F - \frac{1}{6} k^2(F_{tt} + F_tF_U + FF_U^2 + 2FF_U + F^2F_{UU}) + O(k^3)
\]
must be used as the basis for the correction of Eq. (5) in order to obtain a third-order method. Such a correction results in
\[
\frac{dU}{dt} = F + \frac{1}{2} k(F_f + FF_U) + \frac{1}{6} k^2(F_{tt} + F_tF_U + FF_U^2 + 2FF_U + F^2F_{UU}),
\]
which can be discretized by means of the Euler forward difference method to obtain a third-order differencescheme whose finite difference equation is omitted. The characteristic polynomial for the linear stability of the third-order method is
\[
\zeta - 1 = \lambda k + \frac{1}{2} (\lambda k)^2 + \frac{1}{6} (\lambda k)^3,
\]
which shows that this scheme is strongly stable. As illustrated in Fig. 1, its A-stability star is greater than those of the methods previously presented in this section.

The procedure presented in the previous paragraphs can be extended to obtain of fourth-order numerical method by correcting the truncation error terms in the modified equation of the Euler forward method applied to Eq. (8), i.e.,
\[
\frac{dU}{dt} = F - \frac{1}{24} k^3(F_{tt} + F_tF_U + F_tF_U^2 + FF_U^3 + 3F_tF_U + 5FF_U^2F_U + 3FF_U + 3FF_U + 4F^2F_U + 3F^2F_{UU} + FF_{UU} + O(k^4)).
\]
By subtracting the third-order term in the truncation error of this equation from Eq. (8), one can easily obtain the following differential equation for the next correction,
\[
\frac{dU}{dt} = F + \frac{1}{2} k(F_f + FF_U) + \frac{1}{6} k^2(F_{tt} + F_tF_U + FF_U^2 + 2FF_U + F^2F_{UU})
\]
\[+ \frac{1}{24} k^3(F_{tt} + F_tF_U + F_tF_U^2 + FF_U^3 + 3F_tF_U + 5FF_U^2F_U + 3FF_U + 3FF_U + 4F^2F_U + 3F^2F_{UU} + F^2F_{UUU}),
\]
which again, by using the Euler method, yields a fourth-order direct-correction method based on the modified equation. The stability polynomial of this scheme is
\[
\zeta - 1 = \lambda k + \frac{1}{2} (\lambda k)^2 + \frac{1}{6} (\lambda k)^3 + \frac{1}{24} (\lambda k)^4,
\]
which indicates that the fourth-order technique is strongly stable. The A-stability diagram of this method is presented in Fig. 1.
The modified equation of the Euler forward method applied to Eq. (11) is

\[
\frac{dU}{dt} = F - \frac{1}{120} k^4 (F_{uut} + F_t F_{u}^3 + FF_{u}^4 + 4F_t F_{uu} + 8FF_{uu}^2 + 4FF_{uut} + 3F_t^2 F_{uu} + (4FF_t + 12F_t^2 F_{u})F_{uu} + 4F^3 F_{uu}^2
\]
\[\quad + F_{u}(F_{u} + 9FF_{u} + 11F^2 F_{uu}) + 6F^2 F_{uut}
\]
\[\quad + F_t(6F_{uut} + 12FF_{uu} + 6F^2 F_{uu}) + F_{u}F_{uut}
\]
\[\quad + F_{u}(9FF_{uu} + F_t(7F_{uut} + 13FF_{uu}) + 15F^2 F_{uut})
\]
\[\quad + F_{u}(7F^3 F_{uu} + 4F^3 F_{uu} + F^4 F_{uu})].
\]

(13)

By subtracting the fourth-order term in the truncation error of this equation from Eq. (8), the differential equation for the next correction is

\[
\frac{dU}{dt} = F + \frac{1}{2} k(F_t + FF_{u}) + \frac{1}{6} k^2 (F_{u} + F_t F_{u} + FF_{u}^2 + 2FF_{u} + F^2 F_{uu})
\]
\[\quad + \frac{1}{24} k^3 (F_{u} + F_t F_{u} + F_t F_{u}^2 + F_t F_{u}^3 + 3F_t F_{u} + 5FF_t F_{u} + 3FF_{uu})
\]
\[\quad + 3F_t F_{uu} + 4F^2 F_{u} F_{uu} + 3F^2 F_{uut} + F^3 F_{uuu})
\]
\[\quad + \frac{1}{120} k^4 (F_{u} + F_t F_{u}^3 + FF_{u}^4 + 4F_t F_{u} + 8FF_{uu}^2 + 4FF_{uut}
\]
\[\quad + 3F_t^2 F_{uu} + (4FF_t + 12F_t^2 F_{u})F_{uu} + 4F^3 F_{uu}^2
\]
\[\quad + F_{u}(F_{u} + 9FF_{u} + 11F^2 F_{uu}) + 6F^2 F_{uut}
\]
\[\quad + F_t(6F_{uut} + 12FF_{uu} + 6F^2 F_{uu}) + F_{u}F_{uut}
\]
\[\quad + F_{u}(9FF_{uu} + F_t(7F_{uut} + 13FF_{uu}) + 15F^2 F_{uut})
\]
\[\quad + F_{u}(7F^3 F_{uu} + 4F^3 F_{uu} + F^4 F_{uuu})],
\]

(14)

which can be solved numerically by means of the Euler forward method to yield the fifth-order direct-correction method. This scheme is strongly stable and its A-stability diagram is presented in Fig. 1.

As a final example of the correction method briefly described in previous paragraphs, we shall consider a sixth-order method. The modified equation for the fifth-order scheme obtained from Eq. (14) is

\[
\frac{dU}{dt} = F - \frac{1}{720} k^3 (F_{uut} + F_t F_{u}^3 + FF_{u}^4 + 2F_t (15F_{uu} + 15FF_{uu})
\]
\[\quad + F_{u}(10F_{uut} + 20FF_{uu} + 10F^2 F_{uu})
\]
\[\quad + F_{u}(5F_{uut} + 35FF_{uu} + 55F^2 F_{uut} + 25F^3 F_{uu})
\]
\[\quad + F_{u}(50F^2 F_{uu} + 32F^3 F_{uu} + F_{uu}(5FF_{uu})
\]


By subtracting the fifth-order term in the truncation error of this equation from Eq. (14), the differential equation for the sixth-order correction may be easily obtained. Moreover, if this equation is discretized by means of the Euler forward method, it can be easily seen that the sixth-order numerical correction is strongly stable.

Fig. 1 shows the absolute stability diagrams of the five explicit direct-correction methods presented in this section, and indicates that the area of A-stability of these methods increases with the order of the methods, thus showing the good stability properties of the direct-correction methods based on the modified equation. Fig. 1 is in marked contrast with those of linear multistep methods whose stability region decreases as the order of these methods is increased.

3. Asymptotic successive-correction method for the modified equation

In this section, the asymptotic successive-correction technique developed in Parts II [2] and III [1] for the numerical correction of the truncation error terms in the first and the second, respectively, equivalent equations is applied to the third modified or (simply) modified equation. This technique yields a sequence of successive-correction problems where the original finite difference equation is the Euler equation, and a sequence of difference correction equations is introduced in order to obtain new finite difference schemes of as high order as desired. Here, we shall only consider explicit asymptotic successive-correction techniques.

As in Parts II [2] and III [1], we shall consider the standard asymptotic expansion of the global error of the Euler forward method [4] which, for a differential equation which is sufficiently differentiable and a sufficiently small step size such that the Euler method is stable, is

\[ u(t) = U(t) + kU_1(t) + k^2U_2(t) + O(k^3), \]


\[ + F_{UU}(25F_2F_{UUt} + 35F_3F_{UUt} + 15F^4F_{UUU}) \]
\[ + 10F_{UU}F_1^2F_{UUt} + F_1F_{UU}(10F_{Ut} + 50F_{UUt}) \]
\[ + F_1(15F_{Ut} + 10F_{UUt} + 35F^2F_{UUt}) \]
\[ + F_1(30F_{UU}F_{UUt} + 30F_2F_{UUU} + 10F^3F_{UUUU}) \]
\[ + F_1(F_{UUt} + 9F_{Ut}F_{Ut} + 33F_2F_{Ut} + 14FF_{UU}) \]
\[ + F_1(13F_2F_{UU} + F_{UU}(19F_{Ut} + 77F^2F_{Ut})) \]
\[ + F_1(34F_3F_{UU} + 36F_2F_{UUU} + 34F^3F_{UUU}) \]
\[ + F_1F_1(16F_{Ut} + 62FF_{UU} + 46F^2F_{UU}) \]
\[ + 11F_1F^4F_{UUUU} + 5F^4F_{UUUU} + F^2F_{UUUU} + O(k^6). \]
where $U(t)$ is the solution to Eq. (3), and $U_i(t)$ are sufficiently differentiable higher-order corrections to the analytical continuation of the numerical solution (also called correct-value functions [4]) to be determined hereafter.

Introducing Eq. (16) into Eq. (1) yields

$$
U_t + k U_{1t} + k^2 U_{2t} + k^3 U_{3t} + k^4 U_{4t} + O(k^5)
= F + F_U(k U_1 + k^2 U_2 + k^3 U_3 + k^4 U_4)
+ \frac{1}{2} F_{UU}(k^2 U_1^2 + 2k^3 U_1 U_2 + (U_2^2 + 2U_1 U_3)k^4)
+ \frac{1}{3!} F_{UUU}(k^3 U_1^3 + 3k^4 U_1^2 U_2) + \frac{1}{4!} F_{UUUU}(k^4 U_1^4) + O(k^5).
$$

(17)

By subtracting Eq. (3) from Eq. (17) and equating equal powers of $k$, the following differential equation is obtained to leading order,

$$
U_{1t} - F_U U_1 = \frac{1}{2} (F_t + F_t F), \quad U_1(0) = 0.
$$

(18)

The numerical solution to Eq. (18) yields the first correction scheme for Eq. (2). For the numerical solution of Eq. (18), the Euler forward method is used in order that the resulting finite difference equation be a forced version of the linearization of the Euler equation, i.e., Eq. (2); thus, the explicit difference scheme for Eq. (18) reads as

$$
\frac{U_{1t}^{n+1} - U_{1t}^n}{k} - F_U^n U_1^n = \frac{1}{2} (F_t^n + F_t^n F_t^n), \quad U_1^0 = 0,
$$

(19)

where $F^n = F(U^n, t^n)$ and $t^n = nk$. This correction scheme is linearly stable if $\|1 + kF_U(U^n)\| \leq 1$, since then the solution of the Euler equation remains bounded and, therefore, the right-hand-side of Eq. (19) is also bounded.

For the development of higher-order correction equations, the modified equation of the first correction scheme, i.e., Eq. (19), is necessary. This modified equation (multiplied by $k$) is

$$
k U_{1t} = k \left( F_U U_1 + \frac{1}{2} (F_t + F_t F) \right) - \frac{1}{4} k^2 (F_t + 2F_U F + 2F_U^2)
+ 2 U_1 F_U^2 + 2 F_U U_1 + F_U F_{UU} + 2 F_U F_U U + O(k^3).
$$

(20)

By subtracting the second equivalent equations for the Euler and the first correction schemes, i.e., Eqs. (3) and (19), respectively, from Eq. (17), one can easily obtain, to leading order,

$$
U_{2t} - F_U U_2 = \frac{F_{tt}}{6} + F_U^2 \left( \frac{F}{6} + \frac{U_1}{2} \right) + F_{tt} \left( \frac{F}{3} + \frac{U_1}{2} \right)
+ F_{UU} \left( \frac{F^2}{6} + \frac{F_U U_1}{2} + \frac{U_1^2}{2} \right) + \frac{F_t F_t}{6}, \quad U_2(0) = 0.
$$

(21)
This equation can also be solved numerically by means of the Euler method to yield the second correction scheme which is of third order of accuracy.

By using the modified equation for the second correction, i.e., the Euler method applied to Eq. (21), whose expression is omitted for brevity, and the procedure presented in previous paragraphs, i.e., subtracting from Eq. (17) the modified equations for the Euler and the first two corrections schemes, the resulting leading-order equation is

\[ U_{3t} - F_U U_3 = \frac{F_{ut}}{24} + F_U^2 \left( \frac{F_U}{24} + \frac{U_2}{2} \right) + F_U^3 \left( \frac{F_U}{24} + \frac{U_1}{6} \right) \]

\[ + \frac{F_{ut} U_2}{2} + F_{uu} \left( \frac{F_U}{8} + \frac{U_1}{6} \right) \]

\[ + F_U \left( \frac{F_{ut}^2}{8} + \frac{F_{uu} U_1}{8} + \frac{U_1^2}{6} \right) \]

\[ + F_{UU} \left( \frac{F_{ut}^2 U_1}{3} + \frac{U_1^2}{4} \right) \]

\[ + F_U \left( \frac{F_U}{24} + \frac{F_{ut} U_1}{2} + \frac{F_{uu} U_2}{6} \right) \]

\[ + F_{UUU} \left( \frac{F_{ut}^3}{24} + \frac{F_{uu}^2 U_1}{2} + \frac{U_1^3}{6} \right) \]

\[ U_3(0) = 0. \]

This equation can also be solved numerically by means of the Euler method to yield the third correction scheme of fourth-order accuracy.

Further use of the procedure sketched in the previous paragraphs allows for the development of higher-order correction schemes for the Euler equation, and results in numerical methods of order as high as desired. For example, the next correction required to obtain a fifth-order scheme is

\[ U_{4t} - F_U U_4 = \frac{F_{utt}}{120} + \frac{F_{ut}^2}{30} + \frac{F_{u}^2 F_{uu}}{40} + F_u^4 \left( \frac{F_U}{120} + \frac{U_1}{24} \right) \]

\[ + F_{uuu} \left( \frac{F_{u}^3}{30} + \frac{F_{u}^2 U_1}{8} + \frac{U_1^3}{12} \right) \]

\[ + F_{uuu} \left( \frac{F_{u}^2}{20} + \frac{F_{u} U_1}{8} + \frac{U_1^2}{12} \right) \]

\[ + F_{uuu} \left( \frac{F_{u}^4}{120} + \frac{F_{u}^3 U_1}{24} + \frac{F_{u}^2 U_2}{12} + \frac{U_1^3}{4} \right) \]

\[ + F_{uuu} \left( \frac{F_{u}^3}{120} + \frac{F_{u}^2 U_1}{6} + \frac{U_1^2}{12} \right) \]

\[ + F_{u} \left( \frac{F_{u}^3}{120} + \frac{U_2}{6} + \frac{F_{u} U_2}{6} \right) \]

\[ + F_{uu} \left( \frac{F_{u}^3}{20} + \frac{F_{u} U_1}{8} \right) \]

\[ + F_{uu} \left( \frac{F_{u}^4}{120} + \frac{U_1^2}{12} + \frac{F_{u} U_2}{6} \right) \]

\[ + F_{u} \left( \frac{F_{u}^3}{3} + \frac{U_1 U_2}{2} \right) + F_{uu} \left( \frac{F_{u}^2 U_2}{6} + \frac{F_{u} U_2}{2} \right) \]
which can be solved numerically by means of the Euler forward method.

In this paper, the fifth correction was also determined and used to develop a sixth-order scheme, although it is not presented here.

The consistency of the successive-correction method developed in this section is based on the validity of the modified equation as a formal differential equation which exactly solves the Euler forward difference scheme. The technique developed in this section has two advantages over those developed in Parts II [2] and III [1] which are based on the equivalent and the second equivalent equations, respectively. First, all the correction equations in the (third) modified equation are explicit and self-starting. Second, since the truncation error in the modified equation contains no derivatives, this technique does not suffer from the noisy evaluation of the higher-order derivatives that affects the methods based on the equivalent and second equivalent equations. However, the technique presented in this section has a disadvantage over those of Parts II [2] and III [1], i.e., the right-hand-side of the correction equations is highly nonlinear and costly, although its expression can be simplified for autonomous problems.

The linear stability diagrams of the asymptotic successive-correction methods presented in this section can be readily calculated as indicated in Part II [2] and should coincide with that of the Euler forward method; however, their nonlinear stability may be different from that of the Euler method on account of the forcing terms introduced by the successive corrections to the modified equation. Note that, if the time step \( k \) is small enough so that the solution of the Euler method is consistent and stable, i.e., convergent and bounded, then the linear stability condition for the successive-correction equations is the same.

\[ + F_{u tt} \left( \frac{F}{30} + \frac{U_1}{24} \right) + F^2_u \left( \frac{F_{u t}}{120} + F_u \left( \frac{3F}{40} + \frac{U_1}{4} \right) \right) + \frac{F_u U_3}{2} + \frac{F^2_u}{2} \left( F_{u u} \left( \frac{11F^2}{120} + \frac{11FU_1}{24} + \frac{7U_1^2}{12} \right) + \frac{U_3}{2} \right) + \frac{U_2^2}{2} + \frac{FU_3}{2} + U_1 U_3 \right) + F_u \left( \frac{F_{u tt}}{120} + F_u \left( \frac{7F_{u t}}{120} \right) \right) + F_u \left( \frac{13F}{120} + \frac{5U_1}{24} \right) \right) + F_u \left( \frac{F_{u u}^2}{8} + \frac{11FU_1}{24} + \frac{5U_1^2}{12} \right) \right) + F_u \left( \frac{U_1^3}{2} + \frac{FU_2}{2} + F_u \left( \frac{2FU_2}{3} + \frac{3U_1 U_2}{2} \right) \right), \]

\[ U_4(0) = 0, \] (23)
as that of the Euler equation. This consistency and stability indicates that, as $k$ decreases, the asymptotic successive-correction method converges to the solution of the original differential equation.

4. Numerical methods for systems of equations based on the modified equation

The modified equation of the finite difference method for a single, first-order, ordinary differential equation presented in Part I [3] can be readily extended to a system of non-autonomous ordinary differential equations. The direct-correction and asymptotic successive-correction methods developed and studied in the previous sections can also be extended to systems of ordinary differential equations as indicated in the next sections.

4.1. Explicit direct-correction method

We shall consider the system of $N$ differential equations

$$\frac{d}{dt}u_i(t) = F_i(u(t), t), \quad u_i(0) = a_i, \quad i = 1, 2, \ldots, N,$$

(24)

where $u(t) = (u_1(t), u_2(t), \ldots, u_N(t))$, and the Euler forward method for each of the $U_i(t)$

$$\frac{U_i^{n+1} - U_i^n}{k} = F_i(U^n, t^n), \quad U_i^0 = a_i,$$

(25)

where, $i = 1, 2, \ldots, N$ and $U^n = (U_1^n, U_2^n, \ldots, U_N^n)$.

In order to obtain a second-order method from the Euler forward method, the truncation error terms in its modified equation must be corrected. The modified equation corresponding to the Euler forward finite difference scheme is

$$\frac{dU_i}{dt} = F_i - \frac{1}{2}k \left( F_{i,t} + \sum_{j=1}^{N} F_{i,j} F_j \right) + O(k^2), \quad U_i(0) = a_i,$$

(26)

where $F_{i,t}$ denotes partial derivative with respect to time and $F_{i,j}$ is the partial derivative with respect to $U_j$. A second-order numerical method can be obtained by direct correction of this modified equation by truncating its right-hand-side to first-order in the step size and changing the sign of the corresponding term. This yields

$$\frac{dU_i}{dt} = F_i + \frac{1}{2}k \left( F_{i,t} + \sum_{j=1}^{N} F_{i,j} F_j \right), \quad U_i(0) = a_i,$$

(27)

This equation can be approximated by means of the explicit Euler forward method to yield the following second-order numerical method,
where $F_i^n = F_i(U^n, t^n)$. The modified equation of the second-order scheme, i.e., Eq. (28), reads

\[
\frac{dU_i}{dt} = F_i - \frac{1}{6} k^2 \left( F_{i,t,t} + \sum_{j=1}^{N} F_{i,j,j} F_j + \sum_{j=1}^{N} F_{i,j,j}^2 \right)
+ 2 \sum_{j=1}^{N} F_{i,j,j} F_j + 2 \sum_{k>j}^{N} F_{i,j,k} F_j + \sum_{k>j}^{N} F_{i,j,k} F_j + \sum_{k>j}^{N} F_{i,j,k} F_j + O(k^3).
\]

The differential equation that must be solved in order to obtain a third-order numerical method is

\[
\frac{dU_i}{dt} = F_i + \frac{1}{2} k \left( F_{i,t,t} + \sum_{j=1}^{N} F_{i,j,j} F_j \right) + \frac{1}{6} k^2 \left( F_{i,t,t} + \sum_{j=1}^{N} F_{i,j,j} F_j \right)
+ \sum_{j=1}^{N} F_{i,j,j} F_j^2 + 2 \sum_{j=1}^{N} F_{i,j,k} F_j + 2 \sum_{k>j}^{N} F_{i,j,k} F_j + \sum_{k>j}^{N} F_{i,j,k} F_j + O(k^3),
\]

\[U_i(0) = a_i.\]

Following the same procedure, explicit, fourth-order and fifth-order direct-correction methods based on the modified equation can be easily obtained; however, the finite difference equations of these methods are not presented here.

### 4.2. Asymptotic successive-correction method

If both the step size is sufficiently small and the solution of Eq. (24) is sufficiently differentiable, the global error of the Euler scheme, cf. Eq. (25), can be written as

\[
u_i(t) = U_i(t) + kU_{i1}(t) + k^2 U_{i2}(t) + O(k^3),
\]

where $U_i(t)$ is the solution of the second equivalent equation for the Euler forward method and $U_{ij}(t)$ are higher-order corrections to each numerical solution.
Introduction of Eq. (32) into Eq. (24) yields
\[
\frac{dU_i}{dt} + k \frac{dU_{i1}}{dr} + k^3 \frac{dU_{i2}}{dr} + O(k^3) = F_i + \sum_{j=1}^{N} F_{i,j}(kU_{j1} + k^2 U_{j2}) + \frac{1}{2} k^2 \left( \sum_{j=1}^{N} F_{i,j} U_{j1}^2 + 2 \sum_{j>k=1}^{N} F_{i,j,k} U_{j1} U_{k1} \right) + O(k^3),
\]
where \( U = (U_1, U_2, \ldots, U_N) \).

In order to obtain the first correction equation using the asymptotic successive-correction technique developed in Part II [2] and III [1], the modified equation of the Euler forward method, cf. Eq. (26), is first subtracted from Eq. (33) and equal powers of \( k \) are then equated. The result of these manipulations is, at leading order,
\[
\frac{dU_{i1}}{dt} = \sum_{j=1}^{N} F_{i,j} \left( U_{j1} + \frac{F_j}{2} \right) + \frac{1}{2} F_{i,t}, \quad U_{i1}(0) = 0,
\]
which is the differential equation for the first correction scheme. In order to obtain a second-order method, this equation may be discretized by means of the Euler method as
\[
U_{i1}^{n+1} - U_{i1}^n = \sum_{j=1}^{N} F_{i,j} \left( U_{j1}^n + \frac{F_j^n}{2} \right) + \frac{1}{2} F_{i,t}^n, \quad U_{i1}(0) = 0.
\]
Subtraction of the modified equations for the Euler and the first correction schemes from Eq. (33) yields
\[
U_{i2} = \sum_{j=1}^{N} F_{i,j} \left( U_{j1} + \frac{F_j}{6} \right) + \sum_{j=1}^{N} F_{j,t} \left( U_{j1}^n \right) + \frac{1}{2} \sum_{j=1}^{N} F_{i,j} \left( U_{j1} \right) - \sum_{j=1}^{N} F_{i,j} \left( \frac{U_{k1} + F_k}{2} \right),
\]
where
\[
U_{i2}(0) = 0.
\]
The second correction equation may be readily obtained by solving numerically Eq. (36) by means of the Euler forward scheme.

Following the same procedure, the third and fourth correction equations which yield fourth- and fifth-order numerical methods, respectively, can be easily deduced, although their equations are omitted here.

5. Presentation of results for ordinary differential equations

For the sake of convenience, the Euler forward method and the explicit direct-correction methods developed in Section 2, cf. Eqs. (2), (5), (8), (11) and
(14) and the sixth-order method, whose expression has been omitted above, are referred to as E3E0, E3E1, E3E2, E3E3, E3E4 and E3E5, respectively; and, the second-, third-, fourth-, fifth- and sixth-order accurate successive-correction methods presented in Section 3 are referred to as E3C1, E3C2, E2C3, E2C4 and E2C5, respectively. Note that E3C0 coincides with E3E0.

The direct-correction and asymptotic successive-correction methods presented in the previous sections have been used to obtain the numerical solution of problems P0–P8 presented in Parts II [2] and III [1], and

P9: \( F(u, t) = \sin(t)u^8 \) and \( a = 1/2 \) (exact solution: \( u(t) = (121 + 7\cos(t))^{-1/7} \). Computations for \( t \in [0, 100] \)).

P10: \( F(u, t) = (3t - t^2)u \) and \( a = 1 \) (exact solution: \( u(t) = \exp(((9 - 2t)t^2)/6) \). Computations for \( t \in [0, 10] \)).

P11: \( F(u, t) = (3t - t^2)u^2 \) and \( a = 0.1 \) (exact solution: \( u(t) = 1/(10 - 3t^2/2 + t^3/3) \). Computations for \( t \in [0, 10] \)).

Table 1 shows the numerical orders calculated by using the equation presented in Part II [2] for the linear problems P0(+) and P0(−). This table shows that the numerical order approaches, as \( k \) decreases, the theoretical (asymptotic) order for all the direct-correction and successive-correction methods. However, for very small time steps or higher-order methods, round-off errors dominate the truncation errors and there is a loss of numerical order as shown in Table 1 for the E3E5 and E3C5 methods.

Table 2 shows the maximum and \( L^2 \)-norm absolute errors for the difference between the numerical and the exact solutions for the linear problems P0(+)

### Table 1

<table>
<thead>
<tr>
<th>Numerical order</th>
<th>P0(−)</th>
<th>P0(+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E3E0</td>
<td>1.0193</td>
<td>1.0095</td>
</tr>
<tr>
<td>E3E1</td>
<td>2.0342</td>
<td>2.0170</td>
</tr>
<tr>
<td>E3E2</td>
<td>3.0361</td>
<td>3.0180</td>
</tr>
<tr>
<td>E3E3</td>
<td>4.0376</td>
<td>4.0188</td>
</tr>
<tr>
<td>E3E4</td>
<td>5.0387</td>
<td>5.0193</td>
</tr>
<tr>
<td>E3E5</td>
<td>6.0367</td>
<td>6.1304</td>
</tr>
<tr>
<td>E3C1</td>
<td>2.0597</td>
<td>2.0302</td>
</tr>
<tr>
<td>E3C2</td>
<td>3.0564</td>
<td>3.0276</td>
</tr>
<tr>
<td>E3C3</td>
<td>4.0641</td>
<td>4.0314</td>
</tr>
<tr>
<td>E3C4</td>
<td>5.0919</td>
<td>5.0448</td>
</tr>
<tr>
<td>E3C5</td>
<td>6.0981</td>
<td>6.0476</td>
</tr>
<tr>
<td>EM</td>
<td>2.0342</td>
<td>2.0170</td>
</tr>
<tr>
<td>RK</td>
<td>4.0376</td>
<td>4.0188</td>
</tr>
</tbody>
</table>
and P0(−) and indicates that the errors decrease as the order of the numerical scheme increases. This table also shows that the successive-correction methods are more accurate than the direct-correction ones for problem P0(−) but less so for problem P0(+). For linear problems with constant coefficients, the results of Runge–Kutta methods coincide with those of direct-correction techniques, i.e., E3E1 and E3E3 are identical to EM and RK, respectively, as can be shown easily using Taylor series.

A study of the evolution of the errors for the linear problems P0(+) and P0(−) as a function of the step size indicates that the error of the direct-correction techniques and the successive-correction methods decreases as \( k \) decreases until round-off errors begin to dominate; however, the lower-order successive-correction methods are more accurate than the higher-order ones for large \( k \) [5]. This behaviour of the asymptotic successive-correction technique is related to the need for small \( k \) in order that the asymptotic expansion of the global error be accurate. For a fixed order of accuracy, the direct-correction techniques are always more accurate than the successive-correction ones. The most accurate technique is E3E5.

The numerical order for the problems P1–P11 is similar to that for P0(+) and P0(−) [5]. Table 3 shows the numerical orders for problems P1(−) and P4(+), and indicates that the E3E1 and E3E3 methods are not identical to EM and RK, respectively, for nonlinear problems. The results presented in this table and those for all the other nonlinear problems show that the numerical order approaches, as \( k \) decreases, the theoretical (asymptotic) order for all the methods used in this paper.
Tables 4–7 show the maximum, $E_{\text{max}}$, and $L^2$, $E_{L^2}$, norms of the absolute (in some cases, relative) errors for the difference between the numerical and the exact solutions for the linear and nonlinear problems P1–P11. From these tables and others not shown here [5], it is observed that the errors decrease as the order of the numerical methods is increased except for the problems whose solutions blow up. Fig. 2 shows the evolution of the maximum error as $k$ is varied for problem P1(−) for all the methods used in this paper. The results presented in Fig. 2 for P1(−) are representative of the results obtained for the other nonlinear problems P1–P8, whose details are omitted here for brevity [5].

Fig. 2 and Tables 4–7 indicate that, when $k$ is small enough, the methods developed in this paper can be ordered from larger to smaller error as $E_{3E0}$, $E_{3E1}$, $E_{3E2}$, $E_{3E3}$, $E_{3E4}$, $E_{3E5}$, $E_{3C1}$, $E_{3C2}$, $E_{3C3}$, $E_{3C4}$, $E_{3C5}$, $EM$, $RK$. However, when $k$ is very small, roundoff errors control the global error and this ranking suffers some changes.

For a reasonable numerical solution, the time step must be smaller than the inverse of the maximum value of the Jacobian of the nonlinearity, cf. $k \ll O(1/\max |F(U)|)$. For such step sizes, Tables 4–7 and further results not shown here [5], indicate that, for a given order of accuracy, the direct-correction methods are more accurate than the RK scheme and the asymptotic successive-correction techniques developed in this paper, although the magnitude of the error is on the same order.

The complexity of the numerical methods studied in this paper has also been analyzed by counting the number of floating point operations, and the results show that, unless the exact solution is known in explicit form, some numerical

### Table 3

Numerical order for problems P1(−) and P4(+) using the maximum error and the time steps 0.0625, 0.03125, 0.015625 and 0.0078125

<table>
<thead>
<tr>
<th>Numerical order</th>
<th>P1(−)</th>
<th>P4(+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{3E0}$</td>
<td>1.0535</td>
<td>1.0071</td>
</tr>
<tr>
<td>$E_{3E1}$</td>
<td>2.1168</td>
<td>1.9997</td>
</tr>
<tr>
<td>$E_{3E2}$</td>
<td>3.1411</td>
<td>3.0218</td>
</tr>
<tr>
<td>$E_{3E3}$</td>
<td>4.1723</td>
<td>3.9889</td>
</tr>
<tr>
<td>$E_{3E4}$</td>
<td>5.1992</td>
<td>5.0143</td>
</tr>
<tr>
<td>$E_{3E5}$</td>
<td>6.2315</td>
<td>5.9699</td>
</tr>
<tr>
<td>$E_{3C1}$</td>
<td>2.1757</td>
<td>2.0135</td>
</tr>
<tr>
<td>$E_{3C2}$</td>
<td>3.3194</td>
<td>3.0141</td>
</tr>
<tr>
<td>$E_{3C3}$</td>
<td>4.6535</td>
<td>4.0270</td>
</tr>
<tr>
<td>$E_{3C4}$</td>
<td>5.9391</td>
<td>5.0406</td>
</tr>
<tr>
<td>$E_{3C5}$</td>
<td>6.9391</td>
<td>6.0453</td>
</tr>
<tr>
<td>$EM$</td>
<td>2.0557</td>
<td>2.0111</td>
</tr>
<tr>
<td>$RK$</td>
<td>4.0271</td>
<td>4.0140</td>
</tr>
</tbody>
</table>
Table 4
Maximum and $L^2$-norm absolute errors for $\text{P1}(-)$ with $k = 0.0625$, and $\text{P1}(+)$ and $\text{P2}(+)$ both with $k = 0.00625$

<table>
<thead>
<tr>
<th></th>
<th>$E_{\text{max}}$</th>
<th>$E_{L^2}$</th>
<th></th>
<th>$E_{\text{max}}$</th>
<th>$E_{L^2}$</th>
<th></th>
<th>$E_{\text{max}}$</th>
<th>$E_{L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E3E0</td>
<td>$2.4703 \times 10^{-2}$</td>
<td>$1.0091 \times 10^{-1}$</td>
<td>$1.2034$</td>
<td>$2.5363$</td>
<td>$4.6781 \times 10^{-3}$</td>
<td>$4.5160 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3E1</td>
<td>$1.9819 \times 10^{-3}$</td>
<td>$7.2520 \times 10^{-3}$</td>
<td>$3.8752 \times 10^{-2}$</td>
<td>$7.1098 \times 10^{-2}$</td>
<td>$9.9532 \times 10^{-5}$</td>
<td>$7.8682 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3E2</td>
<td>$1.5723 \times 10^{-4}$</td>
<td>$5.2634 \times 10^{-4}$</td>
<td>$1.4248 \times 10^{-3}$</td>
<td>$2.3420 \times 10^{-3}$</td>
<td>$3.417 \times 10^{-6}$</td>
<td>$1.6390 \times 10^{-5}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3E3</td>
<td>$1.2890 \times 10^{-5}$</td>
<td>$4.0444 \times 10^{-5}$</td>
<td>$6.2272 \times 10^{-5}$</td>
<td>$9.563 \times 10^{-5}$</td>
<td>$5.8905 \times 10^{-8}$</td>
<td>$3.7983 \times 10^{-7}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3E4</td>
<td>$1.0754 \times 10^{-6}$</td>
<td>$3.2132 \times 10^{-6}$</td>
<td>$3.0233 \times 10^{-6}$</td>
<td>$4.2357 \times 10^{-6}$</td>
<td>$1.5472 \times 10^{-9}$</td>
<td>$9.4046 \times 10^{-9}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3E5</td>
<td>$9.1360 \times 10^{-8}$</td>
<td>$2.6143 \times 10^{-7}$</td>
<td>$1.5641 \times 10^{-7}$</td>
<td>$2.0746 \times 10^{-7}$</td>
<td>$4.1949 \times 10^{-11}$</td>
<td>$2.4347 \times 10^{-10}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3C1</td>
<td>$1.2810 \times 10^{-3}$</td>
<td>$3.1762 \times 10^{-3}$</td>
<td>$1.9308 \times 10^{-1}$</td>
<td>$3.3652 \times 10^{-1}$</td>
<td>$6.2592 \times 10^{-5}$</td>
<td>$3.0857 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3C2</td>
<td>$8.7480 \times 10^{-5}$</td>
<td>$1.6129 \times 10^{-4}$</td>
<td>$3.3751 \times 10^{-2}$</td>
<td>$5.2267 \times 10^{-2}$</td>
<td>$1.1535 \times 10^{-6}$</td>
<td>$3.9546 \times 10^{-6}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3C3</td>
<td>$7.8877 \times 10^{-6}$</td>
<td>$1.0476 \times 10^{-5}$</td>
<td>$6.1803 \times 10^{-3}$</td>
<td>$8.8120 \times 10^{-3}$</td>
<td>$2.5054 \times 10^{-8}$</td>
<td>$6.7638 \times 10^{-8}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3C4</td>
<td>$7.1123 \times 10^{-7}$</td>
<td>$7.9931 \times 10^{-7}$</td>
<td>$1.1658 \times 10^{-3}$</td>
<td>$1.5629 \times 10^{-3}$</td>
<td>$6.0288 \times 10^{-10}$</td>
<td>$1.3682 \times 10^{-9}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3C5</td>
<td>$6.4130 \times 10^{-8}$</td>
<td>$6.7392 \times 10^{-8}$</td>
<td>$2.2452 \times 10^{-4}$</td>
<td>$2.8696 \times 10^{-4}$</td>
<td>$1.5496 \times 10^{-11}$</td>
<td>$3.0861 \times 10^{-11}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EM</td>
<td>$1.0193 \times 10^{-3}$</td>
<td>$3.8869 \times 10^{-3}$</td>
<td>$2.0355 \times 10^{-2}$</td>
<td>$3.7248 \times 10^{-2}$</td>
<td>$4.1750 \times 10^{-5}$</td>
<td>$3.3879 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RK</td>
<td>$6.6301 \times 10^{-7}$</td>
<td>$2.2511 \times 10^{-6}$</td>
<td>$3.0461 \times 10^{-6}$</td>
<td>$4.5441 \times 10^{-6}$</td>
<td>$7.5748 \times 10^{-10}$</td>
<td>$5.2459 \times 10^{-9}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
methods can be more efficient than the exact solution (numerically calculated by Newton’s method) [5]. As expected, as the order of the method increases, the complexity also increases. Both EM and RK schemes are more efficient than the second- and fourth-order, respectively, direct-correction and successive-correction methods. Direct-correction methods are more efficient than successive-correction ones of the same order.

Fig. 3 shows the maximum error as a function of time for the direct-correction, successive-correction and the Runge–Kutta numerical methods for problems P6a, P6b and P6c. This figure (top) indicates that the direct-correction methods are first-order accurate for problems P6a and P6b, and second-order accurate for problem P6c, while the EM and RK methods are first- and first-order accurate, respectively, for P6a, third- and fourth-order accurate, respectively, for P6b, and second- and fourth-order accurate accurate, respectively, for P6c. Fig. 3 (bottom) shows that the successive-correction methods retain their asymptotic order and the good behaviour of their error even for discontinuous nonlinearities, and indicates the usefulness of these methods for problems with complex nonlinearities. The results presented in Fig. 3 (bottom) are in marked contrast with those of the successive-correction techniques based on the equivalent and the second equivalent equations [2, 1].

We next compare the results presented in this section with those presented in Part I [2] for the successive-correction methods based on the first modified or equivalent equation, cf. the backward EF1–EF4, the centered EC2–EC4 and the all-centered AC1–AC4 asymptotic successive-correction methods, and in Part II [1] for direct-correction and asymptotic successive-correction methods based on the second equivalent equation, cf. the explicit E2E1–E2E4 and the

<table>
<thead>
<tr>
<th></th>
<th>P5(+)</th>
<th></th>
<th>P5(−)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_{\infty}$</td>
<td>$E_{L_2}$</td>
<td>$E_{\infty}$</td>
</tr>
<tr>
<td>E3E0</td>
<td>$5.7397 \times 10^{-3}$</td>
<td>$6.1693 \times 10^{-2}$</td>
<td>$5.6033 \times 10^{-3}$</td>
</tr>
<tr>
<td>E3E1</td>
<td>$4.7851 \times 10^{-5}$</td>
<td>$5.5048 \times 10^{-4}$</td>
<td>$3.4220 \times 10^{-5}$</td>
</tr>
<tr>
<td>E3E2</td>
<td>$6.5767 \times 10^{-7}$</td>
<td>$7.1914 \times 10^{-6}$</td>
<td>$3.6312 \times 10^{-7}$</td>
</tr>
<tr>
<td>E3E3</td>
<td>$1.1259 \times 10^{-8}$</td>
<td>$1.1690 \times 10^{-7}$</td>
<td>$5.2702 \times 10^{-9}$</td>
</tr>
<tr>
<td>E3E4</td>
<td>$2.1868 \times 10^{-10}$</td>
<td>$2.1791 \times 10^{-9}$</td>
<td>$9.1390 \times 10^{-11}$</td>
</tr>
<tr>
<td>E3E5</td>
<td>$4.6125 \times 10^{-12}$</td>
<td>$4.4538 \times 10^{-11}$</td>
<td>$1.7689 \times 10^{-12}$</td>
</tr>
<tr>
<td>E3C1</td>
<td>$3.5044 \times 10^{-5}$</td>
<td>$3.7979 \times 10^{-4}$</td>
<td>$8.3256 \times 10^{-5}$</td>
</tr>
<tr>
<td>E3C2</td>
<td>$4.2451 \times 10^{-7}$</td>
<td>$3.8444 \times 10^{-6}$</td>
<td>$1.9769 \times 10^{-6}$</td>
</tr>
<tr>
<td>E3C3</td>
<td>$6.8003 \times 10^{-9}$</td>
<td>$5.3899 \times 10^{-8}$</td>
<td>$5.8535 \times 10^{-8}$</td>
</tr>
<tr>
<td>E3C4</td>
<td>$1.2713 \times 10^{-10}$</td>
<td>$9.2306 \times 10^{-10}$</td>
<td>$1.9636 \times 10^{-9}$</td>
</tr>
<tr>
<td>E3C5</td>
<td>$2.6230 \times 10^{-12}$</td>
<td>$1.7940 \times 10^{-11}$</td>
<td>$7.1377 \times 10^{-11}$</td>
</tr>
<tr>
<td>EM</td>
<td>$1.2904 \times 10^{-7}$</td>
<td>$1.4120 \times 10^{-6}$</td>
<td>$7.3847 \times 10^{-8}$</td>
</tr>
<tr>
<td>RK</td>
<td>$2.6132 \times 10^{-10}$</td>
<td>$2.7154 \times 10^{-9}$</td>
<td>$1.2843 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
Table 6
Maximum and $L^2$-norm relative errors for P7 with $k = 7.8125 \times 10^{-4}$, and P8 and P9 both with $k = 7.8125 \times 10^{-3}$

<table>
<thead>
<tr>
<th></th>
<th>P7</th>
<th>P8</th>
<th>P9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{Rel}E_{\text{max}}$</td>
<td>$\text{Rel}E_{L2}$</td>
<td>$\text{E}_{\text{max}}$</td>
</tr>
<tr>
<td>E3E0</td>
<td>$7.7723 \times 10^{-2}$</td>
<td>$4.3020 \times 10^{-1}$</td>
<td>$1.8738 \times 10^{-1}$</td>
</tr>
<tr>
<td>E3E1</td>
<td>$5.6852 \times 10^{-4}$</td>
<td>$2.4565 \times 10^{-1}$</td>
<td>$4.9987 \times 10^{-5}$</td>
</tr>
<tr>
<td>E3E2</td>
<td>$4.6827 \times 10^{-6}$</td>
<td>$1.7203 \times 10^{-5}$</td>
<td>$3.9763 \times 10^{-7}$</td>
</tr>
<tr>
<td>E3E3</td>
<td>$4.1126 \times 10^{-8}$</td>
<td>$1.3399 \times 10^{-7}$</td>
<td>$1.0830 \times 10^{-9}$</td>
</tr>
<tr>
<td>E3E4</td>
<td>$3.7617 \times 10^{-10}$</td>
<td>$1.1148 \times 10^{-9}$</td>
<td>$5.2642 \times 10^{-12}$</td>
</tr>
<tr>
<td>E3E5</td>
<td>$3.5225 \times 10^{-12}$</td>
<td>$9.7010 \times 10^{-12}$</td>
<td>$2.3981 \times 10^{-14}$</td>
</tr>
<tr>
<td>E3C1</td>
<td>$4.3079 \times 10^{-4}$</td>
<td>$1.8637 \times 10^{-3}$</td>
<td>$3.0594 \times 10^{-3}$</td>
</tr>
<tr>
<td>E3C2</td>
<td>$3.5204 \times 10^{-6}$</td>
<td>$1.2921 \times 10^{-5}$</td>
<td>$3.3063 \times 10^{-5}$</td>
</tr>
<tr>
<td>E3C3</td>
<td>$3.5994 \times 10^{-8}$</td>
<td>$1.1720 \times 10^{-7}$</td>
<td>$2.6141 \times 10^{-7}$</td>
</tr>
<tr>
<td>E3C4</td>
<td>$3.4716 \times 10^{-10}$</td>
<td>$1.0287 \times 10^{-9}$</td>
<td>$1.5715 \times 10^{-9}$</td>
</tr>
<tr>
<td>E3C5</td>
<td>$3.1779 \times 10^{-12}$</td>
<td>$8.7591 \times 10^{-12}$</td>
<td>$7.3053 \times 10^{-12}$</td>
</tr>
<tr>
<td>EM</td>
<td>$2.8661 \times 10^{-4}$</td>
<td>$1.2368 \times 10^{-3}$</td>
<td>$1.7590 \times 10^{-4}$</td>
</tr>
<tr>
<td>RK</td>
<td>$3.1060 \times 10^{-11}$</td>
<td>$1.0003 \times 10^{-10}$</td>
<td>$2.2732 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
implicit E2I1–E2I4 direct-corrections, and the all-backward E2B1–E2B4 and all-centered E2C3, E2C4 asymptotic successive-correction methods. First, re-call that the EF and EC methods of Part II [2] used an inconsistent starting procedure which reduces the order of the asymptotic fifth-order methods by one, but the AC, E2B and E2C methods are consistently started, so the fifth-order AC method preserves its asymptotic order. Since the E3E and E3C methods require no starting procedure, they incur no loss of order when the nonlinearity is sufficiently differentiable. For problems P6a, P6b and P6c, only the asymptotic successive-correction techniques E2B, E2C and E3C preserve their correct asymptotic order.

<table>
<thead>
<tr>
<th></th>
<th>E_max</th>
<th>E_{L2}</th>
<th></th>
<th>E_max</th>
<th>E_{L2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>E3E0</td>
<td>1.9278 × 10^1</td>
<td>7.7254 × 10^1</td>
<td>E3E0</td>
<td>2.9465 × 10^{-3}</td>
<td>1.4095 × 10^{-2}</td>
</tr>
<tr>
<td>E3E1</td>
<td>1.6311 × 10^6</td>
<td>1.6548 × 10^6</td>
<td>E3E1</td>
<td>1.1251 × 10^{-4}</td>
<td>4.4469 × 10^{-4}</td>
</tr>
<tr>
<td>E3E2</td>
<td>2.4063 × 10^{-2}</td>
<td>8.9993 × 10^{-2}</td>
<td>E3E2</td>
<td>1.3545 × 10^{-6}</td>
<td>5.8779 × 10^{-6}</td>
</tr>
<tr>
<td>E3E3</td>
<td>3.0513 × 10^{-4}</td>
<td>1.0450 × 10^{-3}</td>
<td>E3E3</td>
<td>4.5544 × 10^{-8}</td>
<td>2.2332 × 10^{-7}</td>
</tr>
<tr>
<td>E3E4</td>
<td>1.6614 × 10^{-5}</td>
<td>5.3015 × 10^{-5}</td>
<td>E3E4</td>
<td>1.6448 × 10^{-9}</td>
<td>5.6682 × 10^{-9}</td>
</tr>
<tr>
<td>E3E5</td>
<td>1.1212 × 10^{-6}</td>
<td>3.2612 × 10^{-6}</td>
<td>E3E5</td>
<td>5.3726 × 10^{-11}</td>
<td>1.5753 × 10^{-10}</td>
</tr>
<tr>
<td>E3C1</td>
<td>2.6464</td>
<td>1.0566 × 10^1</td>
<td>E3C1</td>
<td>7.6221 × 10^{-5}</td>
<td>2.8268 × 10^{-4}</td>
</tr>
<tr>
<td>E3C2</td>
<td>2.8568 × 10^{-1}</td>
<td>1.1479</td>
<td>E3C2</td>
<td>2.1661 × 10^{-6}</td>
<td>8.1127 × 10^{-6}</td>
</tr>
<tr>
<td>E3C3</td>
<td>2.5965 × 10^{-2}</td>
<td>1.0509 × 10^{-1}</td>
<td>E3C3</td>
<td>7.7462 × 10^{-8}</td>
<td>2.9637 × 10^{-7}</td>
</tr>
<tr>
<td>E3C4</td>
<td>2.0716 × 10^{-3}</td>
<td>8.4519 × 10^{-3}</td>
<td>E3C4</td>
<td>2.5288 × 10^{-9}</td>
<td>9.4670 × 10^{-9}</td>
</tr>
<tr>
<td>E3C5</td>
<td>1.4837 × 10^{-4}</td>
<td>6.1617 × 10^{-4}</td>
<td>E3C5</td>
<td>1.2009 × 10^{-10}</td>
<td>3.4455 × 10^{-10}</td>
</tr>
<tr>
<td>EM</td>
<td>3.1948 × 10^7</td>
<td>3.2390 × 10^7</td>
<td>EM</td>
<td>8.7402 × 10^{-5}</td>
<td>4.4110 × 10^{-4}</td>
</tr>
<tr>
<td>RK</td>
<td>6.7372 × 10^{-4}</td>
<td>3.0644 × 10^{-3}</td>
<td>RK</td>
<td>6.6551 × 10^{-9}</td>
<td>3.4244 × 10^{-8}</td>
</tr>
</tbody>
</table>

Table 7  
Maximum and $L^2$-norm absolute errors for P10 and P11 both with $k = 0.0625$

Fig. 2. Absolute errors for problem P1(−) as functions of $k$.  

The tables of the errors for the methods presented in this paper and the ones presented in Parts II [2] and III [1] show that the successive-correction techniques and the direct-correction method developed in this paper are more accurate than the successive-correction techniques developed in Parts II [2] and III [1], and the explicit and implicit direct-correction techniques developed in Part III [1] for problems P1–P7. The dependence of the error on both the time step and time is the most useful characteristic of the techniques developed in this paper compared with that of the those developed in previous papers.

Although the mathematical expression of the nonlinear truncation error terms of the modified equation is more complex than the ones in the first and second equivalent equations, the direct-correction and the successive-correction techniques developed in this paper require a smaller number of floating point operations than the ones presented in Parts II [2] and III [1]. The explicit direct-correction methods are the most efficient techniques among those de-
developed in this paper, have better stability properties than the original Euler method, and are even more efficient than Runge–Kutta methods. The computational cost for the successive-correction techniques developed in this paper are slightly costlier than the explicit, and less complex than the implicit, direct-correction methods developed in Part III [1]; however, they are computationally less costly than the successive-correction techniques developed in Parts II [2] and III [1].

The derivation of the successive-correction techniques developed in this paper for non-autonomous problems can be simplified for autonomous ones because the derivatives of the non-linearity in the right-hand side of the modified equation can be simplified in that case, thus reducing the computational cost. Therefore, these techniques are more costly for non-autonomous problems than for autonomous ones. However, the above referred simplification implies a greater reduction of the cost than that associated with the direct-correction and asymptotic successive-correction techniques based on the second equivalent equation for autonomous problems [1].

6. Presentation of results for systems of ordinary differential equations

The second-order ordinary differential equations and systems of ordinary differential equations considered in Parts II [2] and III [1] have been used to assess the accuracy of the numerical techniques developed in Section 4.

Table 8 shows the numerical orders for the direct-correction and the successive-correction methods developed in this paper when applied to problems

<table>
<thead>
<tr>
<th>$k_0$</th>
<th>Numerical order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1</td>
</tr>
<tr>
<td></td>
<td>$1.25 \times 10^{-3}$</td>
</tr>
<tr>
<td>E3E0</td>
<td>1.0397</td>
</tr>
<tr>
<td>E3E1</td>
<td>2.0693</td>
</tr>
<tr>
<td>E3E2</td>
<td>3.0723</td>
</tr>
<tr>
<td>E3E4</td>
<td>5.0774</td>
</tr>
<tr>
<td>E3C1</td>
<td>2.1229</td>
</tr>
<tr>
<td>E3C2</td>
<td>3.1177</td>
</tr>
<tr>
<td>E3C3</td>
<td>4.1355</td>
</tr>
<tr>
<td>E3C4</td>
<td>5.1924</td>
</tr>
<tr>
<td>EM</td>
<td>2.0693</td>
</tr>
</tbody>
</table>
S1, S7 and S9. This table indicates that the numerical order of the direct-correction (E3E) methods coincides with the asymptotic order of these techniques for nearly all the problems studied in this paper. However, the numerical order of the higher-order successive-correction methods, i.e., E3C2–E3C4, is smaller than their asymptotic one for some problems, cf. S9 in Table 8.

Tables 9–11 show the maximum and $L^2$-norm absolute errors for the E3E, E3C and Runge–Kutta methods. For linear problems with constant coefficients, the E3E1 and E3E3 methods coincide with the EM and RK ones, respectively, i.e., for S1 and S2, and nearly coincide with them for S3, S4 and S5. Some of the systems of equations considered in this paper are stiff and linearly unstable; therefore, the Euler forward scheme is not absolutely stable for these problems, and the instability propagates to the asymptotic successive-correction techniques [5]. In particular, the E3C methods behave poorly for nearly all the problems. The most accurate method, amongst those studied in this section, is the E3E4, followed by E3C4, E3E3, RK, E3C2, E3E2, E3C1, E3E1, EM and E3E0, when the E3C methods are stable.

From Tables 8–11 and further numerical tests for methods E2E and E2C [5], it has been observed that the behaviour of the numerical order and the absolute errors of the direct-correction techniques used in this paper when applied to systems of equations, is very similar to the one presented in Section 5 for single equations; however, this is not so for the asymptotic successive-correction techniques which behave better for single equations than for systems of equations. This is to be expected because the Euler forward method for most of the systems of equations considered here is linearly unstable. Furthermore, it should be noted that one rarely would use an explicit method when solving stiff problems because of the extremely small step sizes that would be required for stability reasons.

A comparison between the results presented here and those of Parts II [2] and III [1] clearly indicates that (a) the (third) modified equation offers some computational advantages over those based on the equivalent and second equivalent equations, (b) the instability of stiff and linearly unstable systems of ordinary differential equations appears earlier when using the (third) modified equation than when using the equivalent and second equivalent equations when asymptotic successive-correction techniques are employed (of course, these techniques also become unstable eventually for stiff and linearly unstable systems), (c) the appearance of instability is accelerated by increasing the step size for the stiff and linearly unstable systems studied in this paper as well as in Parts II and III, (d) E2F are more accurate than E3C for problems S3a, S3b, S4, S5, S6 and S8, while the opposite is true for S7, S10 and S11, for (the same) small time steps (for large step sizes, E3C may be more accurate than E2F) until numerical instabilities set in, and (e) E3E is more accurate than E3C which, in turn, is more accurate than E2E and E2F for systems of ordinary
Table 9
Maximum and $L^2$-norm absolute errors for S1, S2 and S3a with $k = 1.5625 \times 10^{-4}$, $7.8125 \times 10^{-5}$ and $3.125 \times 10^{-5}$, respectively

<table>
<thead>
<tr>
<th></th>
<th>$E_{max}$</th>
<th>$E_{l2}$</th>
<th>$E_{max}$</th>
<th>$E_{l2}$</th>
<th>$E_{max}$</th>
<th>$E_{l2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>1.7657 x 10^{-3}</td>
<td>2.0506 x 10^{-2}</td>
<td>8.7095 x 10^{-4}</td>
<td>1.3496 x 10^{-2}</td>
<td>3.0495 x 10^{-1}</td>
<td>8.6294 x 10^{4}</td>
</tr>
<tr>
<td>E3E0</td>
<td>9.2360 x 10^{-6}</td>
<td>1.0042 x 10^{-4}</td>
<td>2.2962 x 10^{-6}</td>
<td>3.5135 x 10^{-5}</td>
<td>1.3997 x 10^{-1}</td>
<td>3.9601</td>
</tr>
<tr>
<td>E3E1</td>
<td>3.6105 x 10^{-8}</td>
<td>3.9258 x 10^{-7}</td>
<td>4.3509 x 10^{-9}</td>
<td>6.9329 x 10^{-8}</td>
<td>9.8542 x 10^{-6}</td>
<td>2.7881 x 10^{-4}</td>
</tr>
<tr>
<td>E3E2</td>
<td>1.259 x 10^{-9}</td>
<td>1.2274 x 10^{-9}</td>
<td>7.1509 x 10^{-12}</td>
<td>1.0941 x 10^{-10}</td>
<td>5.0931 x 10^{-7}</td>
<td>1.4410 x 10^{-5}</td>
</tr>
<tr>
<td>E3E3</td>
<td>2.9399 x 10^{-13}</td>
<td>3.1950 x 10^{-12}</td>
<td>9.2149 x 10^{-15}</td>
<td>1.8703 x 10^{-13}</td>
<td>5.0517 x 10^{-7}</td>
<td>1.4293 x 10^{-5}</td>
</tr>
<tr>
<td>E3E4</td>
<td>4.1812 x 10^{-6}</td>
<td>6.6317 x 10^{-5}</td>
<td>1.1926 x 10^{-6}</td>
<td>2.3220 x 10^{-5}</td>
<td>1.8231 x 10^{-1}</td>
<td>5.0260 x 10^{2}</td>
</tr>
<tr>
<td>E3C1</td>
<td>2.9385 x 10^{-8}</td>
<td>3.1136 x 10^{-7}</td>
<td>2.3101 x 10^{-6}</td>
<td>3.7656 x 10^{-5}</td>
<td>1.7809 x 10^{-2}</td>
<td>5.0396 x 10^{3}</td>
</tr>
<tr>
<td>E3C2</td>
<td>2.2335 x 10^{-10}</td>
<td>1.5553 x 10^{-9}</td>
<td>2.3079 x 10^{-6}</td>
<td>3.7659 x 10^{-5}</td>
<td>1.7926 x 10^{-2}</td>
<td>5.0719 x 10^{3}</td>
</tr>
<tr>
<td>E3C3</td>
<td>6.9922 x 10^{-13}</td>
<td>8.0305 x 10^{-12}</td>
<td>6.8095 x 10^{-6}</td>
<td>5.0540 x 10^{-4}</td>
<td>6.7594 x 10^{-7}</td>
<td>1.0953 x 10^{9}</td>
</tr>
<tr>
<td>EM</td>
<td>9.2360 x 10^{-6}</td>
<td>1.0042 x 10^{-4}</td>
<td>2.2962 x 10^{-6}</td>
<td>3.5135 x 10^{-5}</td>
<td>1.0287 x 10^{0}</td>
<td>2.9105 x 10^{1}</td>
</tr>
<tr>
<td>RK</td>
<td>1.2289 x 10^{-10}</td>
<td>1.2274 x 10^{-9}</td>
<td>7.1511 x 10^{-12}</td>
<td>1.0942 x 10^{-10}</td>
<td>5.0831 x 10^{-7}</td>
<td>1.4381 x 10^{-5}</td>
</tr>
</tbody>
</table>
Table 10
Maximum and $L^2$-norm absolute errors for S3b with $k = 3.125 \times 10^{-5}$, and S4 and S5 both with $k = 3.125 \times 10^{-4}$

<table>
<thead>
<tr>
<th></th>
<th>S3b</th>
<th></th>
<th>S4</th>
<th></th>
<th>S5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_{max}$</td>
<td>$E_{L2}$</td>
<td>$E_{max}$</td>
<td>$E_{L2}$</td>
<td>$E_{max}$</td>
<td>$E_{L2}$</td>
</tr>
<tr>
<td>E3E0</td>
<td>$3.0495 \times 10^4$</td>
<td>$8.6294 \times 10^4$</td>
<td>$6.1408 \times 10^{-2}$</td>
<td>$7.5923 \times 10^{-1}$</td>
<td>$2.1287 \times 10^{-1}$</td>
<td>$7.1713$</td>
</tr>
<tr>
<td>E3E1</td>
<td>$1.3996 \times 10^{-1}$</td>
<td>$3.9600$</td>
<td>$6.4836 \times 10^{-5}$</td>
<td>$8.0129 \times 10^{-4}$</td>
<td>$1.6351 \times 10^{-3}$</td>
<td>$4.6883 \times 10^{-2}$</td>
</tr>
<tr>
<td>E3E2</td>
<td>$9.4556 \times 10^{-6}$</td>
<td>$2.6753 \times 10^{-4}$</td>
<td>$2.4390 \times 10^{-7}$</td>
<td>$3.1060 \times 10^{-6}$</td>
<td>$1.2774 \times 10^{-5}$</td>
<td>$3.6682 \times 10^{-4}$</td>
</tr>
<tr>
<td>E3E3</td>
<td>$2.0186 \times 10^{-7}$</td>
<td>$5.7111 \times 10^{-6}$</td>
<td>$1.8227 \times 10^{-8}$</td>
<td>$2.3091 \times 10^{-7}$</td>
<td>$8.0061 \times 10^{-8}$</td>
<td>$2.2958 \times 10^{-6}$</td>
</tr>
<tr>
<td>E3E4</td>
<td>$1.6350 \times 10^{-7}$</td>
<td>$4.6258 \times 10^{-6}$</td>
<td>$1.4803 \times 10^{-7}$</td>
<td>$1.8753 \times 10^{-6}$</td>
<td>$4.1684 \times 10^{-10}$</td>
<td>$1.1972 \times 10^{-8}$</td>
</tr>
<tr>
<td>E3C1</td>
<td>$1.8231 \times 10^1$</td>
<td>$5.0260 \times 10^2$</td>
<td>$1.0153 \times 10^{-3}$</td>
<td>$1.2276 \times 10^{-2}$</td>
<td>$1.7817 \times 10^{-1}$</td>
<td>$4.6163$</td>
</tr>
<tr>
<td>E3C2</td>
<td>$1.7809 \times 10^2$</td>
<td>$5.0397 \times 10^3$</td>
<td>$2.1049 \times 10^3$</td>
<td>$2.6687 \times 10^4$</td>
<td>$2.8486 \times 10^{-1}$</td>
<td>$7.9644$</td>
</tr>
<tr>
<td>E3C3</td>
<td>$1.7926 \times 10^2$</td>
<td>$5.0719 \times 10^3$</td>
<td>$2.1705 \times 10^3$</td>
<td>$2.7498 \times 10^4$</td>
<td>$1.0106 \times 10^{-1}$</td>
<td>$2.7381$</td>
</tr>
<tr>
<td>E3C4</td>
<td>$6.7550 \times 10^{-7}$</td>
<td>$1.0946 \times 10^9$</td>
<td>$2.6602 \times 10^3$</td>
<td>$3.0713 \times 10^4$</td>
<td>$1.0226 \times 10^5$</td>
<td>$2.4914 \times 10^6$</td>
</tr>
<tr>
<td>EM</td>
<td>$1.0287$</td>
<td>$2.9106 \times 10^3$</td>
<td>$6.4836 \times 10^{-5}$</td>
<td>$8.0129 \times 10^{-4}$</td>
<td>$1.6318 \times 10^{-3}$</td>
<td>$4.6789 \times 10^{-2}$</td>
</tr>
<tr>
<td>RK</td>
<td>$2.2954 \times 10^{-7}$</td>
<td>$6.4945 \times 10^{-6}$</td>
<td>$1.9122 \times 10^{-8}$</td>
<td>$2.4224 \times 10^{-7}$</td>
<td>$7.9716 \times 10^{-8}$</td>
<td>$2.2858 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
differential equations for which the Euler forward method is linearly stable, e.g., S1, because the methods presented in this paper are self-starting and do not contain as higher-order derivatives in the truncation error terms as both the direct-correction and the asymptotic successive-correction techniques based on the first and second equivalent equations for the Euler forward method. It should be recalled that E2E was started with a Runge–Kutta technique and used backward finite difference expressions for the approximation of the higher-order derivatives, whereas E2F is self-starting and approximates the higher-order derivatives by means of forward differences whose stencils increase as the order of these methods increases. Further studies regarding the effects of the time step on the numerical solution of the systems of ordinary differential equations considered in this paper can be found in Ref. [5].

7. Conclusions

The modified equation method has been studied as a means for the development of new numerical techniques for initial-value problems of ordinary differential equations based on the third modified or (simply) modified equation for the improvement of the Euler forward method. Both direct-correction and asymptotic successive-correction methods have been developed. In the direct-correction methods, each higher-order derivative in the truncation error terms has been numerically approximated (with opposite sign) by a finite difference formula. After each correction is obtained, the modified equation of the new difference scheme must be calculated for the next one, and, since no de-
Derivatives of the dependent variable appear in the truncation error terms, the problems associated with the evaluation of higher-order derivatives found in the previous papers of this series do not arise.

All the direct-correction numerical techniques developed in this paper are completely explicit and strongly stable, and have stability diagrams whose area is larger than that of the Euler forward method; furthermore, their stability area increases as the order of the method increases. For linear problems with constant coefficients, the second- and fourth-order methods developed in this paper coincide with the second- and fourth-order accurate Runge–Kutta schemes, respectively.

Asymptotic successive-correction techniques based on the modified equation yield higher-order, conditionally stable numerical schemes which are completely explicit, self-starting and of as high order of consistency as desired. These techniques also have good linear stability properties.

The direct-correction and asymptotic successive-correction numerical methods presented in this paper have been applied to autonomous and non-autonomous, first-order, ordinary differential equations and compared with second- and fourth-order accurate Runge–Kutta schemes. It has been found that the most accurate techniques are those of sixth-order followed by those of fifth-order direct-correction and successive-correction methods which are, in turn, more accurate than the fourth-order Runge–Kutta scheme; the latter is, in turn, more accurate than the fourth-order direct-correction and successive-correction methods. Direct-corrections techniques are, in most of the cases considered here, more accurate than the asymptotic successive-correction ones. It has also been found that the numerical methods based on the modified equation have their expected asymptotic order, except in some special cases.

The methods presented in this paper have also been extended to systems of ordinary differential equations or higher-order differential problems and compared with Runge–Kutta techniques. Second-order autonomous and non-autonomous, linear and non-linear, stiff and non-stiff problems have been studied. It was found that, for a fixed order, the Runge–Kutta technique is more accurate than the direct-correction and successive-correction techniques based on the modified equation of the Euler forward scheme. It was also found that direct-correction schemes are more accurate than asymptotic successive-correction techniques for those systems of ordinary differential equations whose discretization by means of the Euler forward method is linearly stable; for those problems which result in unstable techniques, the asymptotic successive-correction techniques presented here exhibit their instability sooner than similar methods based on the first and second equivalent equations.

The numerical results presented here illustrate the validity of the method of modified equations based on the third modified or (simply) modified equation of two-level finite difference methods for initial-value problems in ordinary
differential equations whose discretization by means of the Euler forward technique results in linearly stable schemes.

Acknowledgements

The research reported in this paper was supported by Project PB94-1494 from the D.G.I.C.Y.T. of Spain.

References


