Optical properties of conics: a method for obtaining reflecting and focusing profiles

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Received 19 April 1999; received in revised form 8 June 1999; accepted 9 June 1999

Abstract

An original method is proposed for determining suitable profiles for two different problems: to produce a given family of wave fronts from a single point source after reflecting on it, and to focus the given wave front on a fixed point. The method uses the optical properties of conics to obtain both reflecting and focusing profiles as the envelope of a specific family of conic sections. It also offers a procedure for determining the caustic of the wave front. © 1999 Elsevier Science B.V. All rights reserved.

PACS: 03.40.Kf; 42.15.Gs; 43.20.+g; 62.30.+d

Keywords: Reflecting-profile; Focusing profile; Wave front; Caustic; Conic section

1. Introduction

The study of optical caustics and the design of reflecting and focusing profiles deserve considerable attention because of their significant technical applications (see Refs. [1–7]).

In the context of geometrical optics with a constant refraction index, we will consider the wave fronts corresponding to a normal rectilinear congruence, i.e. the family of surfaces which orthogonally cut each straight line of the congruence (see Refs. [8,9]).

Suppose we have data corresponding to one of these wave fronts and a point outside it. In this paper we deal with two different problems.

Firstly, suppose that we want to obtain the family of wave fronts from a point source after reflecting on a profile. We solved the problem of determining an appropriate profile by using the optical properties of the conic sections (see Ref. [10]). We will show that it is possible to construct such a profile as the envelope of a specific family of conic sections. In fact, with our construction we obtain several families of suitable reflecting profiles for which we give explicit formulas in terms of the above-mentioned data.

The second problem we deal with is finding a profile such that it focuses the rays emanating from the wave front in a given point \( F \). Again, by using...
the optical properties of conics, we obtain several families of these profiles as envelopes of specific families of conics.

Moreover, we have found an interesting relation between these profiles and the caustic of the given wave front, which also allows us to construct them from the caustic. A different method in the case of reflection was developed in Ref. [11].

In order to simplify the presentation, we have restricted ourselves in Sections 2.1, 2.2 and 2.3 to considering the two-dimensional case, so a wave front becomes a curve instead of a surface. In Section 2.4 we show how the method can be generalized to the 3-dimensional case.

2. Analysis and examples

2.1. Reflecting profiles as envelopes of families of hyperbolas

Let us consider a normal rectilinear congruence of rays in a region of the plane where it does not present singularities. The problem is to find the appropriate shape for a profile such that it produces the initial congruence after reflecting on it from a given point source.

Let \( g \) be one of the wave fronts associated with the normal rectilinear congruence. Suppose we have given a point source \( F \) outside \( g \), so that the distance between \( F \) and \( g \) (i.e. the infimum of the distances between \( F \) and a point of \( g \)) is positive.

For each fixed \( a \) such that \( 0 < 2a < \text{distance}(F, g) \), we will construct a curve \( f_a \) such that the initial congruence coincides with that reflected by it from the given point source \( F \). The construction of \( f_a \) proceeds as follows. For any point \( F' \) in \( g \), consider the hyperbola \( h_{F'}(a) \) which has foci \( F \) and \( F' \), and the distance between its vertices is equal to \( 2a \) (see Figs. 1 and 2). As \( F' \) varies in \( g \), we obtain a family of hyperbolas \( \{h_{F'}(a)\}_{F' \in g} \). Then, \( f_a \) is the envelope (see Ref. [10]) of this family of hyperbolas.

Notice that condition \( 0 < 2a < \text{distance}(F, g) \) is necessary to make sure that the eccentricity \( e = |FF'|/2a \) is greater than 1, as is the case in a hyperbola (see Remark in Section 2.2).

To prove that the constructed curve \( f_a \) is an appropriate profile, let us consider the parametric expression of the wave front \( g \):

\[
g(t) = \left(g_1(t), g_2(t)\right).
\]

where the parameter \( t \) varies in an open real interval.

Then the polar equation of the family of hyperbolas \( h_{F'}(a) \) with respect to the pole \( F \) and polar axis \( FF' \) (see Ref. [10]) is given by:

\[
r(\theta, t) = \frac{|g(t)|^2 - a}{1 + \frac{4a}{2a|g(t)|} \cos \theta},
\]

where \( t \) is the parameter of the family corresponding to \( F' \) (see Fig. 1).
From the optical properties of hyperbolas (see Ref. [10]), we know that the reflected ray at the point \( P \in h_f(a) \) from \( F \) coincides with the ray that, emanating from \( F' \), passes through \( P \).

Let us consider the envelope \( f_a \) of the family formed by the branches closest to the focus \( F \) of the hyperbola \( h_f(a) \). We assert that the normal line to \( g \) at \( F' \) intersects the envelope \( f_a \) at the point \( P(F') \) where the envelope itself is tangent to the hyperbola \( h_f(a) \). In order to simplify the proof of this fact, it is convenient to assume that \( t \) is the polar angle of \( g \) with respect to the \( x \) axis, that is \( g(t) = (\rho(t) \cos t, \rho(t) \sin t) \), where \( \rho(t) = |g(t)| \). Thus, the parametric equation of \( h_f(a) \) is given by \( x(\theta,t) = (r(\theta,t) \cos (\theta + t), r(\theta,t) \sin (\theta + t)) \), and therefore, the condition for a point \( P(\theta,t) \) in \( h_f(a) \) to also be in the envelope \( f_a \) is that \(( \partial x)/(\partial t) = (\partial x)/(\partial \theta) \). In our case, this condition is equivalent to \(( \partial r)/(\partial t)(\theta,t) = (\partial r)/(\partial \theta)(\theta,t) \). which turns into \( \rho(t) r(\theta,t) \sin \theta + \rho'(t) r(\theta,t) \cos \theta = \rho(t) q(t) \). Precisely this is the condition for the vector \( F'P \) to be perpendicular to \( g(t) \), which is equivalent to \( F'P \) being parallel to \( n(t) \), so the assertion holds.

On the other hand, another property that characterizes the hyperbola \( h_f(a) \) is:

\[
\|F'P\| - |FP| = 2a, \tag{3}
\]

for each \( P \in h_f(a) \).

Since the above equation has to be satisfied for all points \( P(F') \) with \( F' \) varying in \( g \), it follows that, for any point \( Q \) on the normal line to \( g \) at \( F' \), the difference between the optical paths \( [F'P(F')Q] \) and \( [FP(F')Q] \) is the same quantity \( 2a \) for all points \( F' \) in \( g \) (see Fig. 1). This means that each wave front produced from the point source \( F \) by reflection on \( f_a \) is optically parallel to \( g \), and therefore, the corresponding normal rectilinear congruence is the given one. Thus, \( f_a \) is a suitable reflecting profile for our problem.

Analogously, the envelope \( f_a' \) of the family formed by the branches closest to focus \( F' \) is another suitable reflecting profile (see Fig. 2).

Notice that when parameter \( a \) varies in such a way that \( 0 < 2a < \text{distance}(F,g) \), we obtain two families \( \{f_a\} \) (see Fig. 3) and \( \{f'_a\} \) that solve our problem.

In order to obtain an explicit equation for the reflecting profile \( f_a \), let us consider the unitary normal vector to \( g \) pointing to the semiplane of \( F, n(t) \).

According to previous observations, \( f_a \) has to verify the following parametric equation:

\[
f_a(t) = g(t) + \lambda n(t). \tag{4}
\]

A straightforward calculus gives

\[
\lambda = \frac{2a^2 - |g(t)|^2}{2a + g(t)n(t)}. \tag{5}
\]

Eqs. (4) and (5) provide a general method to determine the family of reflecting profiles \( \{f_a\} \). We used them to construct the reflecting profiles represented in Figs. 2 and 3.

Of course, the reflecting profile \( f'_a \) has to verify similar equations:

\[
f'_a(t) = g(t) + \lambda' n(t), \tag{4'}
\]

where

\[
\lambda' = \frac{2a^2 - |g(t)|^2}{-2a + g(t)n(t)}. \tag{5'}
\]

Remark: Observe that, although \( g \) has been taken without singularities, the profiles \( f_a \) and \( f'_a \) could
have singularities if the denominator of $\lambda$ or $\lambda'$ vanishes.

2.2. Reflecting and focusing profiles as envelopes of families of ellipses

Let $g$ be a wave front associated with a normal rectilinear congruence, as before. In this case, we will consider the family of ellipses $e_r(a)$, with $F'$ varying in $g$, and where $e_r(a)$ is the ellipse with foci $F$ and $F'$, and with major axis $2a$ (see Fig. 4). Notice that now we have to set $2a > \text{distance}(F,g)$, so that the eccentricity $e = |FF'|/2a$ is less than 1 as correspond in an ellipse.

As in the case of hyperbolas, for each $a$ with $2a > \text{distance}(F,g)$, the envelopes of this family of ellipses are two curves, one, $f_a$, on the left of $F$ and $g$, and the other, $f'_a$, on the right of $F$ and $g$. Analogously to the case of hyperbolas, the parametric equations for $f_a$ and $f'_a$ are given by Eq. (4) and Eq. (4') respectively, where $n(t)$ and $\lambda$ are as above, but now

$$
\lambda' = -\frac{2a^2 - |g(t)|^2}{2 - 2a + g(t)n(t)}. \tag{6}
$$

In this case, as before, the normal line to $g$ at $F'$ intersects the ellipse $e_r(a)$ in two points $Q(F')$, where the ellipse itself is tangent to the envelopes $f_a$ and $f'_a$ respectively (see Fig. 4).

Thus, if the wave front $g$ propagates in the direction of $F$, then $f_a$ is a focusing profile, since the optical path $[F'Q(F')F]$ is constant equal to $2a$ for all $F' \in g$. In the same way, $f'_a$ is a reflecting profile, since for any point $Q$ on the normal line to $g$ at $F'$, the difference between the optical paths $[F'Q]$ and $[FQ(F')Q]$ is the same quantity $2a$ for all points $F'$ in $g$.

Analogously, in the case that $g$ propagates in the opposite direction to $F$ then $f_a$ is a reflecting profile, whereas $f'_a$ is a focusing profile.

Finally, let us observe that the families of ellipses $\{e_r(a)\}_{a < |FF'|/2}$ and hyperbolas $\{h_r(a)\}_{a > |FF'|/2}$ are cofocal with foci $F$ and $F'$ (Fig. 5). Remark: In general, for a given wave front $g$ and a fixed point $F$, if we fix $a > 0$, there could be some regions of $g$ where $d(F,F') < 2a$ and others where $d(F,F') > 2a$. In this case, the profiles should be constructed by considering envelopes of families of hyperbolas in some regions and envelopes of families of ellipses in others.

2.3. Caustics and reflecting profiles

Let us consider a wave front $g$ associated with a normal rectilinear congruence and a fixed point $F$. 

![Fig. 4. Reflecting and focusing profiles $f_a$ and $f'_a$ as the envelopes of a family of ellipses. Here $a = 3$, $F = (0,0)$ and $g(t) = (-6 + 10 \cos t, 9 \sin t)$, $t \in [-\pi/8, \pi/8]$.](image)

![Fig. 5. Cofocal families of hyperbolas $h_r(a)$ and ellipses $e_r(a')$ obtained from the wave front $g(t) = (-5 + 10 \cos t, 10 \sin t)$.](image)
Suppose that $g$ presents a caustic $c$ outside a region containing $F$ and $g$. By definition, $c$ is the envelope of the normal lines to $g$ or, equivalently, the geometrical locus of its centres of curvature (see Ref. [8,9]). Thus, if $g(t)$ is the parametric expression of $g$, $c$ will have to verify the following parametric equation:

$$c(t) = g(t) + \rho_s(t)n(t),$$

where $\rho_s(t)$ is the radii of curvature of $g$ at the point $F' = g(t)$ and $n(t)$ is the unitary normal vector to $g$ in $F'$ pointing to the semiplane of $F$.

Select $a$ with $0 < 2a < \text{distance}(F, g)$. For any point $F'$ on the wave front $g$, consider the centre of curvature of $g$ at $F'$, $F'\prime(F')$, which is on the caustic of $g$. Let $e_{\alpha}(F')$ denote the ellipse with foci $F$ and $F'\prime(F')$ and major semi-axis $d' = (\sqrt{|F'\prime(F')|^2}/2) - a = \rho_s(t) - a$ (see Fig. 6).

For any point $F'$ on the wave front $g$, let us consider the reflecting profile $f_\alpha$ constructed as the envelope of the family formed by the branches closest to focus $F$ of the hyperbolas $h_{\alpha}(a)$ (see Section 2.1).

The next result shows how these curves are related.

The ellipse $e_{\alpha}(F')$ is tangent to the hyperbola $h_{\alpha}(a)$ in the same point where $h_{\alpha}(a)$ is tangent to $f_\alpha$ (see Fig. 6).

In order to demonstrate this result, it is sufficient to show that the intersection between the ellipse $e_{\alpha}(F')$ and the hyperbola $h_{\alpha}(a)$ is the point $P(F')$ in which $f_\alpha$ is tangent to $h_{\alpha}(a)$ (see Section 2.1).

From the properties of conics, we know that if an arbitrary point $P$ is in $e_{\alpha}(F')$, then $P$ satisfies $|F'\prime(F')P| + |FP| = 2d$, and if $P$ is in $h_{\alpha}(a)$, then $P$ satisfies $|F'\prime(F')P| - |FP| = 2a$. It follows that if $P$ is in $e_{\alpha}(F')$ and $h_{\alpha}(a)$ simultaneously, then $|F'\prime(F')P| + |PF| = 2a + 2d = |F'\prime(F')|$. This implies that $P$ is in the normal line to $g$ at $F'$, and therefore $P = P(F')$ (see Fig. 6).

This property allows us to construct the reflecting profile $f_\alpha$ as the envelope of the family of ellipses $\{e_{\alpha}(F')\}_{F' \text{ varying in the wave front } g}$ (see Fig. 7).

On the other hand, since the arc-length parameter $s$ of the caustic is, except for an additive constant, the radii of curvature of the wave front $g$, it follows that, given the caustic $c$, we can determine a suitable reflecting profile simply by parametrizing the caustic by arc-length $s$ and taking the envelope of the family of ellipses $\{e_{\alpha}(s)\}_s$ with foci $F$ and $c(s)$ and with major semi-axis $d = s - s_0$ ($s_0$ constant).

Similarly, we can obtain the reflecting profile $f_\alpha$ for each $a$ with $0 < 2a < \text{distance}(F, g)$ in terms of the caustic $c$. For this consider the ellipses $e_{\alpha}(F')$ with foci $F$ and $F'\prime(F')$ (the centre of curvature of $g$)
at $F'$, and major semiaxis $d = (\|F'F\|/2) + a = \rho_2(t) + a$. Then $f_\rho'$ is the envelope of the family of ellipses $\{e_\rho(F')\}_{\rho}$, $F'$ varying in the wave front $g$.

In a symmetrical way, it is possible to obtain the profiles $f_\rho$ and $f_\rho'$, constructed as the envelopes of the family of ellipses $\{e_\rho(a)\}_{\rho}$ (see Section 2.1), in terms of the caustic $c$ of $g$.

### 2.4. The 3-dimensional case

The results of Sections 2.1, 2.2 and 2.3 can be directly generalized to the 3-dimensional case. We will briefly show how to do this, keeping to the same notation as far as possible.

We start from a given point $F$ and a wave front $g$ that in this case is a surface having the parametric expression

$$g(t) = (g_1(t), g_2(t), g_3(t)).$$

where $t = (t_1, t_2)$ varies in an open domain of the real plane. Let $n(t)$ be the unitary normal vector to $g$ pointing to the semispace of $F$, i.e.

$$n(t) = \pm \left( \frac{\partial g}{\partial t_1} \wedge \frac{\partial g}{\partial t_2} \right),$$

where the sign depends on whether $(\partial g)/(\partial t_1) \wedge (\partial g)/(\partial t_2)$ is pointing to the semispace of $F$ or not.

For each $F'$ in $g$ let us consider the families of surfaces $\{f_\rho\}$ and $\{f_\rho'\}$ given in parametric form by Eqs. (4) and (5) and Eqs. (4') and (5'), respectively. These surfaces are now the envelope of biparameetric families of hyperbolas $h_\rho(a)$, with $2a < \text{distance}(F, g)$, and it follows from Section 2.1 that they are suitable reflecting surfaces for our problem.

We have represented in Fig. 8, for a specific wave front $g$, two of these surfaces $f_\rho, f_\rho'$, as well as the hyperbola $h_\rho(a)$ for a particular point of $g$.

Similarly, identical equations to Eq. (4), Eq. (4'), Eqs. (5) and (6) (see Section 2.2), give rise to the construction of reflecting and focusing surfaces as envelopes of biparameetric families of ellipses.

Fig. 8. Reflecting surfaces $f_\rho, f_\rho'$ ($a = 0.2$), and hyperbola $h_\rho(a)$. Here $g(t) = (t_1, 0.5 + t_1^2 - 3t_1^2/t_2, t_2)$, $F = (0,0,0)$ and $F' = g(t)$ with $t = (0.1,0.1)$.

Finally, let us note that we can construct reflecting and focusing profiles in the 3-dimensional case replacing the above families of hyperbolas and ellipses with the families of hyperboloids and ellipsoids that we get by rotating the above conics around their focal axes.

### Acknowledgements

Partial financial support from DGESIC, Research Grants PB97-1080 (C.C.) and PB97-1095 (N.A.), is acknowledged.

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