Linearly implicit methods for the nonlinear Schrödinger equation in nonhomogeneous media

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Abstract

The nonlinear Schrödinger equation in one-dimensional Cartesian coordinates is studied numerically by means of three linearly implicit finite difference methods that result in block tridiagonal matrices at each time level, as a function of the damping, pumping and changes in the refraction index. The numerical experiments performed to determine the effects of the time step, spatial step size, and implicitness parameters of the dispersion and nonlinear terms show that the most accurate results are obtained when both the dispersion and nonlinear terms are treated by means of a second-order-accurate trapezoidal discretization and full linearization of the nonlinear terms. It is also shown that if either the dispersion or the nonlinear terms are treated with a first-order-accurate approximation, substantial errors are found, and these errors are largest when the nonlinear terms are discretized by means of a first-order difference and do not decrease substantially as the number of grid points is increased. It has been observed that when solitons move from a medium to another one characterized by a higher refraction index, the amplitude and speed of the soliton increase whereas its width decreases. Upon approaching the interface, the amplitude of the soliton was found to increase downstream while some radiation was observed upstream; both the increase in amplitude and radiation were found to increase with the change in the refraction index. The effect of the distance where the index of refraction changes was found to be small if this distance is smaller than the width of the soliton. For solitons propagating from a medium to another one of lower refraction index, the amplitude and speed of the soliton decrease whereas its width increases, the amplitude of the soliton increases downstream and spreads towards the boundary where it is reflected. When the index of refraction

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decreases below a threshold, it was found that the soliton may become trapped at the interface. The results presented in this paper indicate that theories based on treating solitons as particles, perturbation methods, inverse scattering, and variational formulations can provide accurate results only when the change in the refraction index is sufficiently small.

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1. Introduction

The objectives of this paper are severalfold. First, linearly implicit finite difference methods are developed for the study of the focusing, nonlinear Schrödinger equation (NLSE) in one-dimensional Cartesian coordinates and finite domains subject to Dirichlet boundary conditions at both boundaries. These methods have been the subject of numerous studies by the author and co-workers [1,2] who have applied them with great success to one- and two-dimensional reaction–diffusion equations. In this paper, the accuracy of these methods is assessed as a function of the time step, spatial step size or number of grid points, implicitness parameters of the discretization of the dispersion and nonlinear terms, full and approximate diagonal and triangular approximations to the Jacobian matrix, and damping and pumping in the NLSE. These numerical experiments permit us to select the most accurate linearly implicit finite difference equation.

The second objective of this study is to analyze the effects of (local) damping and (local) pumping on the propagation of solitons in finite lines, and to determine the radiation associated with them. The third objective is to analyze the propagation of solitons through media whose refraction index undergoes a smooth change in a localized region as a function of both the jump in refraction index and the width of the region where this index changes. The goal of this study is to determine the propagation of solitons through nonhomogeneous media, the radiation emitted by the soliton upon encountering a change in the refraction index, and the dynamics of the soliton prior to, during and after crossing the region where the refraction index changes. In particular, the conditions under which the soliton becomes trapped at the interface and the validity of perturbation methods, particle approaches, variational principles, etc., are determined as a function of the jump in the refraction index. The fourth and final objective is to study the dynamics of solitons in finite lines where the refraction index is not uniform, upon many collisions of the solitons with the boundaries.

In order to compare the approach followed in this paper with other approaches that have been previously used, a brief review of both numerical
methods for the NLSE and different techniques for dealing with the interaction of solitons with interfaces are reviewed in the following paragraphs.

The NLSE in one spatial dimension and Cartesian coordinates has been studied numerically by means of a large variety of numerical techniques such as finite difference, finite element and spectral methods. Herbst et al. [3] analyzed the NLSE in a finite domain subject to natural boundary conditions and in the absence of forcing, damping, etc., by means of methods of lines based on a finite element (Galerkin) technique with and without mass lumping and (a) the energy-conserving, variable time step, leapfrog technique developed by Sanz-Serna [4], (b) an implicit midpoint rule implemented in a predictor-corrector manner, and (c) the scheme introduced by Delfour et al. [5], i.e., they considered three different types of time discretization, as a function of the parameter that multiplies the nonlinear term, and concluded that, without mass lumping, methods (b) and (c) do not conserve the mass and momentum invariants, whereas the same methods conserve the mass invariant if the mass matrix is lumped. Herbst et al. [3] also showed that the leapfrog technique conserves the mass and momentum invariants if the mass matrix is lumped, although this technique requires very small time steps due to its explicit character. Sanz-Serna [6] obtained optimal $L^2$-norm rates of convergence for numerical methods that employ both finite difference and finite element discretizations in space, while treating the time integration by means of leapfrog and modified Crank–Nicolson procedures. His modified Crank–Nicolson–Galerkin scheme includes both the Delfour et al. [5] and the Strauss–Vázquez [7] methods, and conserves the mass and momentum invariants of the NLSE.

Akrivis et al. [8] employed the Galerkin method in space and two implicit, Crank–Nicolson-type second-order methods for the NLSE without damping and provided $L^2$-norm error bounds of optimal order of accuracy. Their implicit time discretizations are those of Delfour et al. [5] and the midpoint or one-stage Gauss–Legendre implicit Runge–Kutta scheme, while the spatial discretization was based on piecewise linear, continuous elements for Dirichlet boundary conditions and piecewise linear, continuous periodic splines for periodic boundary conditions. Tourigny [9] has provided $H^1$-norm estimates for the backward Euler and Crank–Nicolson discretizations of Galerkin finite element methods applied to the NLSE.

It must be pointed out that most Galerkin finite element methods for the NLSE have treated the spatial discretization of the nonlinear term by means of a product approximation, e.g., [1].

Delfour et al. [5] employed an implicit finite difference method for the NLSE and treated the nonlinear term as suggested by Strauss and Vázquez [7], whereas Shamardan [10] employed three-point compact operators for the discretization of the spatial coordinate and the implicit midpoint rule for the time integration that result in second- and fourth-order accuracy in time and space, respectively. Akrivis [11] employed a Crank–Nicolson discretization and
the Delfour et al. [5] scheme for the nonlinear term, and linearized the resulting nonlinear algebraic equations by means of the Newton method; he also employed an iterative modification of the linear scheme and proved the second-order accuracy of his method. Sanz-Serna and Verwer [12] used a second-order-accurate spatial discretization of the NLSE while time integration was performed by means of (a) the iterative implicit midpoint rule, (b) an implicit method with explicit treatment of the nonlinear terms, (c) an iterative pseudolinear midpoint rule, and (d) two conservative fractional-step methods, and showed that methods (b) and (d) perform rather poorly. They also showed that lack of exact conservation properties may lead to undesirable nonlinear blow-up, e.g., method (a); there are useful numerical schemes which perform in a very stable way and yet do not conserve energy exactly; also, exact conservation does not necessarily guarantee the success of numerical method for the NLSE.

Ablowitz et al. [13] employed second-order-accurate finite difference discretizations of the second-order spatial derivations and treated the nonlinear terms in two different manners, so that the resulting system of coupled, nonlinear ordinary differential equations for the nodal values is either integrable or nonintegrable, and found that the sensitivity of the focusing NLSE to spatial discretization stems from the homoclinic structure associated with it. They also showed that instabilities are avoided by using the integrable discretization of Ablowitz and Ladik [14] although the time discretization may trigger instabilities, and that the integrable scheme breaks down for the nonfocusing NLSE at large nonlinearities. Their studies also suggest that there may be fundamental differences between symplectic and nonsymplectic discretizations of certain Hamiltonian systems.

The integrable discretization of Ablowitz and Ladik [14] generalized the inverse scattering transform (IST) [15] to cover nonlinear partial difference equations. This generalization is very useful for developing numerical schemes which maintain the main properties of the original partial differential equation, and has been implemented and further developed by Taha and Ablowitz, e.g., [16], and Taha [17], who showed that this generalization is much faster than finite difference methods including that of Ablowitz and Ladik [12] and finite Fourier techniques, i.e., pseudospectral methods, [18].

Pseudospectral techniques solve nonlinear evolution equations with periodic boundary conditions by calculating spatial operators in the discrete Fourier space and the nonlinear terms on a discrete grid in space, and can be used to study modulational instabilities. The generalization proposed by Taha [17] is also more accurate than pseudospectral techniques for small amplitudes, whereas, for large ones, a split-Fourier method is superior. Split-step methods factorize the operator equation into a sequence of linear and nonlinear operators which may be integrated sequentially in order to obtain the numerical solution at the next time level. For the NLSE without forcing, damping, etc.,
the nonlinear operator may be integrated analytically if some approximations are made, because this operator is a nonlinear ordinary differential equation, whereas the exponential of the linear operator may be approximated by a rational function and second-order-accurate finite difference methods; alternatively, one may also use Fourier series for the solution at the next time level and thus obtain a split-Fourier method [19].

Recently, a variety of symplectic methods [20] have been developed for the NLSE. For example, Tang et al. [21] have applied two symplectic integrators to a second-order spatial discretization of the NLSE that results in a Hamiltonian system, and showed that these integrators conserve rather well the mass or charge, momentum and energy. It is well known, however, that symplectic integrators may preserve the structure of the phase space but may not conserve the invariants of Hamiltonian systems. Furthermore, in some applications, e.g., mechanical systems, it may be more convenient to preserve the conservation of, say, kinetic energy than the symplectic structure of the corresponding Hamiltonian whereas, in other applications, the converse may be more desirable [14]. On the other hand, symplectic integrators may be the most accurate techniques for long-time integrations, although they may be more time consuming than spectral methods or linearly implicit finite difference techniques [22].

In this paper, the focusing, one-dimensional NLSE in the finite line is analyzed numerically by means of linearly implicit finite difference methods based on the linearization of the algebraic equations obtained from the application of $\theta$-techniques. These techniques have been previously used with great success to study one- and two-dimensional nonlinear reaction–diffusion equations [1,2], and their numerical accuracy is assessed here as a function of the time step, spatial step size, implicitness parameters for the dispersion and nonlinear terms of the NLSE, damping, pumping, and approximations to the Jacobian matrix in the first part of this paper for a medium with constant refractive index. The interaction between incident and reflected solitons and the conservation characteristics of the linearly implicit finite difference methods are also studied in the first part.

The reflection and transmission characteristics of collimated light beams incident on an interface separating two dielectric media, at an angle close to that for total internal reflection, have been the subject of intensive theoretical investigation because such a reflection results in a finite displacement of the reflected beam from its geometric optics path [23]. Aceves et al. [23,24] presented a theory that describes the global and transmission characteristics of a self-focused channel propagating at an oblique angle of incidence to an interface separating two or more self-focusing nonlinear dielectric media, that employs an equivalent particle moving in an equivalent potential. Their theory provides ways to determine the location of each wave packet and its accuracy was assessed by comparing its results with those obtained from the numerical
solution of the NLSE by means of a split-Fourier or beam propagation method.

Aceves et al. [24] also explained beam breakup by examining the decomposition of the channel in one medium into its soliton and radiation (continuous spectrum) components after it has crossed into the new medium. Knapp [25] studied the one-dimensional NLSE with a (small) random potential function to model pulses in very long low-dispersion optical fibers and showed that, if the initial pulse is a soliton solution of the unperturbed NLSE, an accurate equivalent-particle theory may be developed.

Abdullaev et al. [26] have used an adiabatic approximation to analyze the interaction of a spatial soliton with the modulated interface between two different nonlinear media, and found a surface wave. Their adiabatic approximation is based on perturbation theory, assumes that the amplitude of the soliton remains constant, and employs a second-order ordinary differential equation for the location of the soliton center. These approximations can only be justified provided that the jump in the refractive index of small. Abdullaev and Caputo [27] have also used an adiabatic approximation in their studies of soliton propagation through media with spatially modulated dispersion by solving first-order ordinary differential equations for the soliton’s mass, momentum and energy, and showed that the adiabatic approximation is a good one for small velocities. As will be shown below, the ordinary differential equations for the soliton’s mass, momentum and energy do not form a closed set, and closure may be achieved at different levels by imposing that higher-order moments of the NLSE [28] are evaluated by means of the solution corresponding to the unperturbed soliton [26,27], by deriving a new potential based on numerical calculations or physical considerations [23,24] or in an ad hoc manner [29].

Closely related to the adiabatic and equivalent-particle theories are the variational approaches which require the existence of either a Lagrangian or a Hamiltonian of the NLSE. If an invariant exists, then one may assume that the complex amplitude of the soliton may be approximated by means of an analytical function with parameters that depend on time. Substitution of this approximation in the Hamiltonian and minimization of the result provides a set of differential equations for the parameters [30].

Other approximations for dealing with the propagation of solitons through nonhomogeneous media include direct perturbation methods and techniques based on the inverse scattering transform, and numerical analysis. Kodama and Ablowitz [31] have used multiple-scales perturbation or direct perturbation methods to investigate the evolution of solitary waves in the presence of small perturbations, and showed that the leading-order equation corresponds to that of the unperturbed soliton and that the validity of further terms depends on a small parameter that characterizes the difference between the unperturbed and perturbed NLSEs. Their direct perturbation approach is based on a pertur-
bation of the NLSE. By way of contrast, Karpman [32] has developed a perturbation theory for nonlinear waves based on the inverse scattering method and showed that the effects of small perturbations result in a slow change of the soliton parameters, a deformation of the soliton shape, and the formation of soliton tail which is a small-amplitude wave packet of growing length. Lantz et al. [33] have shown numerically that a small Gaussian beam impinging upon a thin nonlinear film can switch from a total reflection state to a transmission state if the nonlinearity is positive.

In the second part of this paper, soliton propagation through a medium whose refraction index undergoes a smooth, finite change over a distance is analyzed as a function of both the jump in the refraction index and the distance over which this jump occurs, by means of linearly implicit methods in order to examine the reflection, transmission and radiation of the soliton upon crossing the interface and their effects upon collision of the soliton with the boundaries of a finite line subject to Dirichlet boundary conditions. A further goal of this study is to assess the validity of adiabatic and equivalent-particle analyses for solitons propagating through nonlinear, nonhomogeneous media.

The paper has been organized as follows. In Section 2, both the governing equations and their finite difference discretizations are presented. A rather lengthy section deals with the presentation of results which consist of the assessment of (a) the accuracy of the linearized implicit methods presented here, (b) the effects of damping and pumping on the soliton propagation, (c) the interaction of solitons with interfaces where the refraction index undergoes a change, and (d) the propagation and bouncing of solitons in finite lines with nonuniform refraction indices.

2. The NLSE in the finite line

In this paper, the following NLSE is considered,

\[
\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2} + \text{ln}(x)|u|^2 + (g|u|^2 - \alpha)u, \tag{1}
\]

where \(i^2 = -1\), \(u\) is a complex value, \(t \in (0, \infty)\) and \(x \in (a, b)\) denote the time and spatial coordinate, respectively, \(n(x)\) is the refraction index, \(\alpha\) is the damping coefficient and \(g\) denotes the Maxwell–Block gain coefficient. Eq. (1) has been studied for \(a = -50\) and \(b = 150\) subject to \(u(t, a) = u(t, b) = 0\), and

\[
u(0, t) = \cosh^{-1} \frac{x}{\sqrt{2}} \exp(i x/2), \tag{2}
\]

\[
u(0, t) = \cosh^{-1} \frac{x - 50}{\sqrt{2}} \exp[-i (x - 50)/2], \tag{3}
\]
\[ u(0,t) = \cosh^{-1} x, \] 

which correspond to the 1-, 2- and \( N \)-soliton solutions, respectively, of the NLSE with \( n(x) = 1 \) and without damping and gain in an infinite line. Eq. (3) corresponds to two identical solitons travelling in opposite directions.

Since \( u = A + iB \), where \( A \) and \( B \) are real, the separation of the real and imaginary parts in Eq. (1) yields the following system of coupled equations:

\[
\frac{\partial A}{\partial t} = -n(x) \frac{\partial^2 B}{\partial x^2} - (A^2 + B^2)B + (g(A^2 + B^2) - x)A, \tag{5}
\]

\[
\frac{\partial B}{\partial t} = n(x) \frac{\partial^2 A}{\partial x^2} + (A^2 + B^2)A + (g(A^2 + B^2) - x)B, \tag{6}
\]

which can be written in a more concise manner as

\[
\frac{\partial U}{\partial t} = D \frac{\partial^2 U}{\partial x^2} + S(U, g, x), \tag{7}
\]

where \( U = (A, B)^T \), \( S \) is a vector, and \( D \) is a matrix whose values can be easily determined from the above equations.

2.1. Linearized, implicit \( \theta \)-methods

In this paper, Eq. (1) was discretized by means of the following \( \theta \)-method,

\[
\Delta U = k(1 - \theta_s)D \frac{\partial^2 U^n}{\partial x^2} + k(1 - \theta_s)S^n + k\theta_s D \frac{\partial^2 U^{n+1}}{\partial x^2} + k\theta_s S^{n+1}, \tag{8}
\]

where \( \Delta U = U^{n+1} - U^n \), \( k \) is the time step, and \( 0 \leq \theta_s \leq 1 \) and \( 0 \leq \theta_s \leq 1 \) denote the implicitness parameters for the dispersion and nonlinear terms, respectively. Eq. (8) is a nonlinear elliptic equation at each time level which, upon the discretization of the spatial derivatives, would result in a nonlinear system of algebraic equations. However, by linearizing the nonlinear terms, one can easily obtain the following linear elliptic equation at each time level,

\[
\Delta U = kD \frac{\partial^2 U^n}{\partial x^2} + kS^n + k\theta_s D \frac{\partial^2 \Delta U}{\partial x^2} + k\theta_s J^n \Delta U, \tag{9}
\]

where \( J \) denotes the Jacobian matrix of the mapping of transformation \( U \to S \) and is a full matrix. If the second-order derivatives in Eq. (9) are discretized as

\[
\frac{\partial^2 U_j}{\partial x^2} = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + O(h^2), \tag{10}
\]

where \( j \) denotes the \( j \)th grid point and \( h \) is the spatial step size, one can easily deduce a block tridiagonal matrix for \( \Delta U \) which may be easily solved by means of LU decomposition. This block tridiagonal matrix has the interesting
property that the blocks off the diagonal are diagonal while that in the diagonal is a full matrix. Since LU decomposition requires the inversion of this full matrix, the calculation of $\Delta U$ may be very time consuming if a large number of grid points are employed in the calculations. For this reason, it may be more efficient to approximate the Jacobian by either diagonal or triangular matrices. If the Jacobian matrix is approximated by a diagonal one whose elements coincide with the corresponding ones of $J$, then the resulting method is referred to as $DL$ and the block of the main diagonal of the block tridiagonal matrix corresponding to Eq. (9) is diagonal.

Since $U$ is a two-dimensional vector, there are two possible approximate triangularizations to the Jacobian matrix: a lower one which corresponds to the linearization of the equation for $A$ with respect to this dependent variable and the linearization of the equation for $B$ with respect to $A$ and $B$, and an upper one which corresponds to the linearization of the equation for $A$ with respect to $A$ and $B$, and the linearization of the equation for $B$ with respect to $B$. These two triangularizations are here referred to as $ABL$ and $BAL$, respectively, whereas the full linearization described above is referred to as $FL$.

### 3. Presentation of results

Fig. 1 shows the propagation of the 1-soliton solution as a function of $t$ and $x$ for $a = -b = -50$, $k = 0.05$, 3201 grid points, $\theta_x = \theta_y = 0.5$, i.e., an $O(k^2 + h^2)$ method, and $g = \alpha = 0$, and indicates that the soliton propagation...
is almost identical to that of a soliton propagating in an infinite domain until it is very close to the right boundary. Upon collision with this boundary, the soliton amplitude increases and the soliton rebounds and propagates towards the left boundary.

3.1. Numerical errors

In order to assess the accuracy of the linearized implicit $\theta$-methods presented in this paper, a numerical study was performed as a function of $k, h, \theta_s, \theta_s, \alpha$ and $g$ for $a = -50$ and $b = 150$. The accuracy was assessed in terms of the mass or charge, $M(t)$, and energy, $E(t)$, of the soliton defined as

$$M = \int_a^b |u|^2 \, dx, \quad E = \int_a^b \left( \frac{\partial u}{\partial x} \right)^2 - \frac{1}{2} |u|^4 \right) \, dx.$$  (11)

These integrals were evaluated numerically by means of the trapezoidal rule with two-point, second-order, central difference discretizations of the first-order derivatives at the interior points, and three-point, second-order differences at the boundaries. Since the initial conditions used in this study were those corresponding to a soliton in an infinite line without gain and damping, and since the initial location of the soliton in the finite line was sufficiently far away from the boundaries, we have defined $mrel = M(t)/M(0)$ and $erel = E(t)/E(0)$ to determine the accuracy of the linearized methods employed in this paper, and some sample results are presented in Figs. 2–9, which correspond, unless otherwise stated, to $h_x = h_s = 0.5, g = a = 0$;

FL: solid line; ABL: dashed line; DL: dashed-dotted line.)

Fig. 2. Mass and energy of the soliton as functions of $t$. ($n(x) = 1, a = -50, b = 150, k = 0.01, 801$ grid points, $\theta_s = \theta_s = 0.5, g = \alpha = 0$; FL: solid line; ABL: dashed line; DL: dashed-dotted line.)
of the boundaries. Fig. 2 also shows that ABL underpredicts both the mass and the energy of the soliton, whereas DL overpredicts the mass and underpredicts the energy. The errors of both DL and ABL increase as k or h are increased; for the same values of the parameters shown in Fig. 2 but with $k = 0.05$ and $0.10$, the decimal logarithms of $m_{rel}$ and $e_{rel}$ are approximately 0.2 and 0.8, respectively, at about $t = 38$ and 18, respectively, whereas, for $k = 0.05$ and 1601 grid points, the decimal logarithms of $m_{rel}$ and $e_{rel}$ are approximately 0.2 and 0.8, respectively, at about $t = 38$. For $k = 0.05$ and 401 grid points, the decimal logarithms of $m_{rel}$ and $e_{rel}$ are approximately 0.2 and 1.0, respectively, at about $t = 38$.

The numerical errors for $\theta_s = 1$ are illustrated in Fig. 3, which indicates that the errors in mass and energy conservation increase as a function of time and that DL is more accurate than FL and ABL; however, the results presented in Fig. 4 show that the implicitness parameter for the numerical discretization of the dispersion terms plays a key role in the conservation of mass and energy.
and that, overall, FL is more accurate than partial linearization methods. Similar results to those shown in Fig. 4 were obtained for $k = 0.01$ and 1601 grid points, and $k = 0.05$ and 801 grid points. On the other hand, the logarithm of the mass and energy, respectively, of the soliton for FL and DL remained nearly constant until about $t = 38$ and then increased steeply to about $10^4$ and $10^6$, respectively, for $k = 0.05$ and 801 grid points, and $k = 0.01$ and 1601 grid points. Therefore, the numerical treatment of the nonlinear terms plays a much more important role on the accuracy of the linearization methods presented in this paper than that of the dispersion ones. This is a consequence of both the errors introduced by the time linearization and/or the approximation of the Jacobian matrix by diagonal or triangular ones. It must be pointed out that the propagation of solitons through homogeneous media with $g = \alpha = 0$ is
Fig. 7. Mass and energy of the soliton as functions of $t$. ($n(x) = 1$, $a = -50$, $b = 150$, $k = 0.01$, 801 grid points, $\theta_s = \theta_i = 0.5$, $g = 0.05$, $x = 0.05$: FL: solid line; ABL: dashed line; DL: dashed-dotted line.)

Fig. 8. Effects of the time step (left) and number of grid points (right) on the mass and energy of the soliton as function of $t$. (FL, $n(x) = 1$; $a = -50$, $b = 150$, $\theta_s = \theta_i = 0.5$, $g = 0$, $x = 0$. Left: 801 grid points; $k = 0.01$: solid line; $k = 0.05$: dashed line; $k = 0.10$: dashed-dotted line; $k = 0.20$: dotted line. Right: $k = 0.05$; 801 grid points: solid line; 1601 grid points: dashed line; 401 grid points: dashed-dotted line; 401 grid points and $k = 0.20$: dotted line.)
characterized by a balance between the steeping associated with the nonlinear terms and the spreading of the dispersion.

Although not shown here, an explicit treatment of either the dispersion \( h_x = 0 \) or the nonlinear \( h_s = 0 \) terms was associated with large errors in the mass and energy of the soliton.

Fig. 5 shows that the effect of the damping losses on the soliton propagation is to cause a decrease in both the mass and energy, and that all the linearization methods presented in this paper provide nearly identical results. The differences between the results of FL, DL and ABL decrease as \( \alpha \) is increased from 0.05 to 0.10. Fig. 6 illustrates the effects of the pumping coefficient on the soliton propagation and indicates that, after an initial transient, FL and ABL reach almost constant values that correspond to larger mass and energy; on the other hand, DL oscillates about higher constant values than the other methods analyzed in this paper. The effects of both the damping and the pumping are shown in Fig. 7, which indicates that, for the values of the parameters con-
sidered in this figure, the damping losses are larger than the gain associated with pumping, and that the three linearization methods considered in this paper provide nearly the same results.

The results presented above indicate that the time step, the spatial step size and the implicitness parameters play a key role in determining the accuracy of the linearized implicit methods presented in this paper, and that FL is more accurate than the other techniques presented here. Therefore, in order to determine approximate steps in both space and time, we have performed extensive studies with FL and some of them are presented in Figs. 8 and 9. Fig. 8 indicates that, for 801 grid points, time steps equal to 0.01 and 0.05 produce nearly identical results, and that the mass and energy of the soliton are not very sensitive to the time step and spatial step size provided that $k \leq 0.05$ and the number of grid points is larger than 401.

Fig. 9 shows that the energy of the soliton is much more sensitive to $\theta_s$ than its mass, and that the most accurate results are obtained with $\theta_s = 0.5$. Fig. 9 also shows that, for $\theta_s = 1$, an increase in the number of grid points does not yield a corresponding increase in accuracy. The results presented in the right side of Fig. 9 clearly indicate that a midpoint evaluation of the nonlinear terms is most accurate and that, for $\theta_s = 1$, an increase in the number of grid points degrades the accuracy of the mass and energy of the soliton because such an increase causes an increase in the accuracy of the discretization of the dispersion terms and, most importantly, a different implicit treatment of the nonlinear and dispersion terms causes an imbalance between them.
The propagation of the 2-soliton solution in a finite line subject to Dirichlet boundary conditions is illustrated in Fig. 10, which shows that the interaction between the counterpropagating solitons occurs at about \( t = 25 \) when the amplitude of the solution is largest; later on, the two solitons emerge from the collision with the same amplitude and speed and propagate freely as if they were in an infinite medium until they interact with and bounce from the boundaries. Such an interaction and bouncing are analogous to those shown in Fig. 1.

The evolution of the \( n \)-soliton solution in a finite line is presented in Fig. 11, which illustrates that the initial amplitude of the solution first decreases, and the soliton radiates energy towards the boundaries. This radiation increases in magnitude as \( t \) increases, and the maximum amplitude of the soliton oscillates on account of the energy radiated and reflected from the boundaries. Similar results to those shown in Fig. 11 have also been observed for \( \frac{250}{c_0} \) and \( t > 70 \), except that the radiation is preceded by the formation of a local relative minimum on both sides of the soliton at \( t = 30 \) and 70. Therefore, the length of the domain plays an important role in the numerical solution of \( n \)-soliton solutions.

### 3.2. Effects of damping and pumping

Fig. 12 shows the propagation of the soliton in the presence of damping, and illustrates both the steep drop in the amplitude of and the spreading of the soliton. These phenomena cause spatial oscillations that reach the boundaries.
These oscillations, however, are damped in time and have disappeared for $t \geq 80$. The damping of the soliton increases as $\alpha$ is increased. Similar results to those shown in Fig. 12 have also been observed for the $n$-soliton solution.

Fig. 13 shows the mass and energy of the soliton as functions of time and the damping and pumping, and indicates that the mass of the soliton decreases as $\alpha$ is increased while it increases as $g$ is increased; however, for the values of the parameters considered in this paper, the mass of the soliton decreases as a function of time.

Fig. 13. Effects of damping and pumping on the mass and energy of the soliton as functions of $t$. (FL, $n(x) = 1$; $a = -50$, $b = 50$, $k = 0.05$, 801 grid points $\theta_i = \theta_s = 0.5$, $g = 0$, $\alpha = 0.05$.)
The results presented in Fig. 13 also show that, with pumping, the energy of the soliton first increases until it reaches a maximum and, then, decreases on account of the damping. The value and location of the energy maximum are functions of $a$ and $g$. Therefore, the results shown in Fig. 12 are also indicative of the parameters of Fig. 13.

3.3. Effects of the refraction coefficient

In order to illustrate the effects of the refraction coefficient on the propagation of solitons governed by the NLSE in finite lines, we have employed the following expression,

$$n(x) = 1 + \beta(1 + \tanh \gamma x),$$

where the constants $\beta$ and $\gamma$ correspond to the jump in and width of the change in refraction index at $x = 0$. Note that $\beta = 0$ corresponds to the refraction index employed in the previous section. Moreover, the values of $\beta$ and $\gamma$ correspond to a change in refraction index from 1 to $1 + 2\beta$ from $x = -\infty$ to $x = \infty$.

Figs. 14 and 15 show the propagation of the 1-soliton solution for $(\beta, \gamma) = (0.1, 1)$ and $(0.5, 1)$, respectively. Both figures indicate that, until $t = 50$, the soliton propagates as if it were in an infinite medium; however,
when it reaches the location where the refraction index changes, there is
an initial increase in the amplitude of the soliton downstream, which is fol-
lowed up by a decrease and some radiation upstream of the soliton as shown
in Fig. 14. Furthermore, the results of Fig. 14 show that the amplitude and
speed of the soliton increase upon crossing the location where the refrac-
tion index increases, whereas its width decreases. Similar results to those ex-
hibited in Fig. 14 have also been observed for $(b; c) = (0.1, 2)$ and $(0.1, 5)$, thus
indicating that the soliton propagation in an inhomogeneous medium with a
sharp increase in refraction index is not very sensitive to the distance in which
this index changes, provided that this distance is smaller than the soliton’s
width.

The results presented in Fig. 15 show that both the amplitude and speed of
the soliton increase upon crossing the location where the refraction index in-
creases, and that the increase in amplitude downstream of the soliton also
increases. A fundamental difference between the results shown in Figs. 14 and
15 is that large changes in the refraction index result in radiation both up-
stream and downstream of the soliton once this has crossed the location where
the change in refraction index occurs. It is interesting to point out that for
$(b, \gamma) = (-0.1, 1), (-0.1, 2)$ and $(-0.1, 5)$, i.e., when the soliton propagates
from a medium to another one with a lower refraction index, it has been ob-
served that the amplitude and speed of the soliton decrease, and radiation can
be seen upstream of the soliton. However, for $(b, \gamma) = (-0.5, 1)$, the results
presented in Fig. 16 indicate that an increase in the soliton’s amplitude occurs upstream rather than downstream (cf. Fig. 15) and that this increase spreads towards the downstream boundary and results in an oscillatory amplitude whose intensity decreases from the downstream boundary. Fig. 16 also shows that no radiation is observed downstream and that the soliton has almost been trapped at the interface.

Fig. 17 shows the average position of the soliton defined as

$$x_{av} = \frac{1}{M} \int_a^b xm \, dx,$$

where $m = |u|^2$ denotes the mass of the soliton per unit length, as a function of time for different values of $(\beta, \gamma)$. This figure illustrates the increase in the soliton’s speed as $\alpha$ is increased as well as the trapping of the soliton at the interface for $(\beta, \gamma) = (-0.5, 1)$. As previously stated the average position of the soliton is nearly independent of $\beta$ for $\alpha = 0.1$ and $-0.1$ and $1 \leq \beta \leq 5$.

It must be noted that Eq. (1) yields

$$\frac{\partial m}{\partial t} = -n(x) \frac{\partial q}{\partial x} + (2gm - x)m,$$
where
\[ iq = u^* \frac{\partial u}{\partial x} - u \frac{\partial u^*}{\partial x}, \]
and \( q \) is the momentum of the soliton per unit length.

Integration of Eq. (14) yields
\[ \frac{dM}{dt} = \int_a^b \left[ 2(gm - x)m + q \frac{\partial n}{\partial x} \right] dx + n(a)q(t,a) - n(b)q(t,b), \]
where \( q(t,a) = q(t,b) = 0 \) for the Dirichlet boundary conditions considered in this paper.

The first moment of Eq. (14) yields
\[ \frac{\partial(mx)}{\partial x} = -n(x) \frac{\partial q}{\partial x} + 2(gm - x)mx, \]
which, upon integration, becomes
\[ \frac{d}{dt} \left( \int_a^b mx \, dx \right) = \int_a^b \left[ 2(gm - x)mx + q \frac{\partial (nx)}{\partial x} \right] \, dx + an(a)q(t,a) - bn(b)q(t,b). \]

Using Eq. (16), it is an easy matter to show that
\[ \frac{dx_{av}}{dt} = \frac{1}{M} \left( \int_a^b \left[ 2(gm - x)m(x - x_{av}) + q \frac{\partial (n(x - x_{av}))}{\partial x} \right] \right) \, dx, \]
which is a first-order ordinary differential equation for the average position of the soliton. This equation, however, is not closed because it depends on both \( M(t) \) and \( q(t,x) \), and \( M(t) \) depends, in turn, on \( q(t,x) \). Therefore, in order to
obtain a closed system of equations for $M(t)$ and $x_{av}$, one must make certain approximations to $q(t,x)$ as indicated in Section 1. Moreover, Eq. (19) may be differentiated with respect to $t$ to yield a second-order ordinary differential equation for the average location of the soliton that resembles Newton’s second law. If $g = \beta = 0$, Eq. (16) can be integrated analytically to yield
\[
M(t) = M(0) \exp(-2\alpha t),
\]
in agreement with the results shown previously (cf. Figs. 12 and 13).

3.4. Effects of the refraction coefficient and the reflection of solitons at the boundaries

The effect of a change in the refraction coefficient in a finite line with Dirichlet boundary conditions is shown in Figs. 18–20. When the refraction coefficient is constant throughout the domain, the soliton propagates as if it were in an infinite medium until it collides with the boundaries. Upon such collision, the amplitude of the soliton increases, and the soliton bounces back and propagates in the opposite direction towards the other boundary as illustrated in Fig. 18. However, when the refraction index is not uniform, the soliton moves from a medium to another one with a different refraction index and, after collision with the upstream boundary, crosses the two media in opposite order; therefore, the soliton would accelerate upon an increase in the refraction coefficient and, after colliding with the boundary, would decelerate upon
crossing again the interface. The effects of several crossings of the interface are illustrated in Figs. 19 and 20. Fig. 19 shows that the amplitude of the soliton undergoes an increase downstream as the soliton crosses the interface where the refraction index increases, and that radiation appears both upstream and downstream; the amplitude of this radiation increases as the soliton bounces.

Fig. 19. Propagation of the 1-soliton solution as a function of $t$ and $x$. ($\textbf{FL}, \beta = 0.3, \gamma = 5; a = -50, b = 150, k = 0.05, 3201$ grid points, $\theta_i = \theta_f = 0.5, g = 0, \alpha = 0$.)

Fig. 20. Propagation of the 1-soliton solution as a function of $t$ and $x$. ($\textbf{FL}, \beta = -0.3, \gamma = 5; a = -50, b = 150, k = 0.05, 3201$ grid points, $\theta_i = \theta_f = 0.5, g = 0, \alpha = 0$.)

crossing again the interface. The effects of several crossings of the interface are illustrated in Figs. 19 and 20. Fig. 19 shows that the amplitude of the soliton undergoes an increase downstream as the soliton crosses the interface where the refraction index increases, and that radiation appears both upstream and downstream; the amplitude of this radiation increases as the soliton bounces.
back from the boundary and moves towards the interface. Upon crossing the interface again, the soliton decelerates and its amplitude decreases, whereas the background radiation increases. This process is repeated and results in a background environment characterized by a large amount of chaotic behavior.

If the soliton moves from a medium with a refraction index to another one with a smaller index, radiation is generated upstream of the soliton. This radiation is reflected from the boundary and interacts with the soliton which bounces back from the boundary and moves towards the interface where the refraction index now increases, thus increasing the soliton’s amplitude and speed. This process is repeated upon further collisions of the soliton with the boundaries as indicated in Fig. 20. A comparison between Figs. 19 and 20 clearly indicates that the short-time behavior of solitons in finite lines is very much dependent upon the propagation direction and the refraction index; for example, the background radiation in Fig. 19 is much smaller than that of Fig. 20.

The average position of the soliton corresponding to the results presented in Figs. 18–20 is shown in Fig. 21. This figure illustrates clearly that the speed of the soliton when it propagates between the boundaries is constant if the refraction index is uniform and changes sign upon the collision of the soliton with the boundaries. When the soliton moves in the direction in which the

Fig. 21. Average position of the 1-soliton solution as a function of \( t \). (\( FL; a = -50, b = 150, k = 0.05, 3201 \) grid points, \( \theta_s = \theta_c = 0.5, g = 0, x = 0 \). Left: \( (\beta, \gamma) = (0.3, 5) \): solid line; \( (-0.3, 5) \): dashed line; \( (0.1, 5) \): dashed-dotted line; \( (0, 0) \): dotted line.)
refraction index increases, the speed of the soliton increases upon encountering the change of medium, remains constant as it propagates towards the boundary and bounces back with equal speed but different direction upon the collision with the boundary. When the reflected soliton encounters again the change in refraction index, its speed decreases, and the process is repeated. However, when the soliton moves in the direction in which the refraction index decreases, the amount of background radiation does not allow definition of the soliton in a coherent and physically meaningful manner, and, in fact, the term soliton may not be used in such a case.

4. Conclusions

Three linearly implicit finite difference equations for the NLSE have been developed. These methods come from the linearization of $\theta$-schemes and result in block tridiagonal matrices for the real and imaginary components of the soliton amplitude. One of these methods results in full matrices, whereas the others involve either diagonal or triangular matrices.

Numerous experiments have been performed to determine the effects of the time step, spatial step size, and implicitness discretization of the dispersion and nonlinear terms in the presence of damping and pumping, and shown that the most accurate results are obtained when both the dispersion and nonlinear terms are treated by means of a second-order-accurate trapezoidal discretization. If either the dispersion or the nonlinear terms are treated with a first-order-accurate, temporal approximation, substantial errors are found; these errors are largest when the nonlinear terms are discretized by means of a first-order-accurate approximation and do not decrease substantially as the number of grid points is increased.

It has also been found that a fully linearized method is more accurate than partially linearized, diagonal or triangular methods, except when there is damping; in this case, the results are almost independent of the linearization if the damping losses are larger than the gain associated with the pumping. The amplitude of the soliton was found to decrease as a function of time when there is damping, and the damping was found to result in background radiation whose intensity decreases as a function of time.

For the cases analyzed in this paper, it was found that the pumping increases initially the mass and energy of the soliton which then decrease because of the damping. The largest magnitude of the mass of the soliton and its temporal location were found to be functions of the damping and pumping.

The propagation of solitons through media whose index of refraction changes smoothly but rapidly was found to be a function of the change in the refraction index and the direction of propagation of the soliton. When solitons move from a medium to another one characterized by a higher refraction
index, the amplitude and speed of the soliton increase whereas its width decreases. The soliton was also found to propagate as if it were in an infinite homogeneous medium when it was located far away from the interface. Upon approaching the interface, the amplitude of the soliton was found to increase downstream while some radiation was observed upstream; both the increase in amplitude and radiation were found to increase with the change in the refraction index. The effect of the distance where the index of refraction changes was found to be small, if this distance is smaller than the width of the soliton.

When the soliton propagates from a medium to another one of lower refraction index, its amplitude and speed decrease whereas its width increases, the amplitude to the soliton increases downstream and spreads towards the boundary where it is reflected. When the index of refraction decreases below a threshold, it was found that the soliton may become trapped at the interface.

In finite lines subject to Dirichlet boundary conditions, it was found that the direction of propagation and the refraction index play a key role in determining the dynamics of solitons. For solitons propagating from a medium to another one of higher refraction index, the soliton preserves its shape longer than when it propagates from a medium to another one of lower refraction index. In either case, it was found that, upon several collisions with the boundaries, there is a large amount of chaotic background radiation associated with the radiation of energy at the interface, its reflection at the boundaries and its further interaction with the soliton.

The results presented in this paper indicate that theories based on treating solitons as particles, perturbation methods, inverse scattering, and variational formulations can provide accurate results only when the change in the refraction index is sufficiently small.

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References