ON DIVISION ALGEBRAS SATISFYING MOUFANG IDENTITIES

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ON DIVISION ALGEBRAS SATISFYING MOUFANG IDENTITIES

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ABSTRACT

Among the algebraic identities satisfied by alternative algebras play a fundamental role the so called Moufang identities. In the first section of this note we characterize division algebras satisfying one of the sided Moufang identities. As a consequence we obtain all the division normed real Moufang sided algebras. Up to isomorphism there are seven algebras in both cases, including the well known alternative algebras R, C, H and O. Moreover, there are three division right Moufang algebras with dimensions 2, 4 and 8 and another three division left Moufang algebras of the same dimensions. Finally we prove that R, C, H and O are the only division normed algebras satisfying the Moufang middle identity.
A nonassociative algebra $A \neq 0$ is said to be a division algebra if for all $x \in A$, $x \neq 0$, the linear operators $L_x, R_x: A \rightarrow A$ given by $L_x: y \mapsto xy$ and $R_x: y \mapsto yx$ are bijective maps. If only the maps $L_x$ ($x \neq 0$) are bijective then $A$ is said to be a left division algebra. In a similar way $A$ is a right division algebra when $R_x$ is bijective for all $x \in A$, $x \neq 0$.

We recall that an alternative algebra $A$ satisfies the following identities:

$$y((xz)x) = ((yx)z)x \quad (1)$$
$$x(yz) = (xy)z \quad (2)$$
$$x(yz) = x(y(zx)) \quad (3)$$

These identities are called right, middle and left Moufang identity respectively.

Let $B$ be an algebra whose product $.$ satisfies the right Moufang identity and $f: B \rightarrow B$ an involutive automorphism. We define a new product

$$yx = f(y)x$$

for any $x, y \in B$. Relative to this product the underlying vector space of $B$ is an algebra that satisfies the right Moufang identity. Indeed, for any $x, y, z \in B$ we have

$$y((xz)x) = y((f(x)z)x) = y((xf(z))x)$$
$$= f(y)((xf(z))x) = ((f(y)x)f(z))x$$
$$= ((yf(x))z)x = ((f(y)x)z)x = ((yx)z)x$$

In particular, if $B$ is an alternative algebra we obtain a new algebra which is right Moufang. Moreover, if $B$ is a division algebra then the new algebra is so. In Sec. 1 we will prove that every division right Moufang algebra is obtained from a division alternative algebra in this way. Similarly it is shown that if $B$ is a division alternative algebra and $f$ an involutive automorphism then the algebra determined by the product

$$xy = xf(y)$$

is a division left Moufang algebra, being all such algebra given in this way.

We recall that a real nonassociative algebra $A$ is a normed algebra when $A$ is normed as a real vector space and $\|xy\| \leq \|x\|\|y\|$ for any $x, y \in A$. The determination of all the division real normed algebras is a very hard problem, even in the case that these algebras are restricted to satisfy
some type of identity. At present it remains still open a Wright conjecture which asserts that every division real normed algebra is finite-dimensional.\cite{1} On the other hand, the existence of one-sided division real normed algebras of infinite dimension with unit element by the same side it has been independently proved in\cite{2,3}.

In the associative case the classical Arens extension of the Gelfand-Mazur theorem asserts that $R$, $C$ and $H$ are the only division normed associative real algebras. This result can be generalized in different ways. So as a consequence of the Bruck and Kleinfeld theorem\cite[Corollary 2]{4} it is obtained that every division normed alternative real algebra is isomorphic to $R$, $C$, $H$ or the division algebra of real octonions $O$. (see also\cite[Theorem 3.2]{5}).

Section 2 of this note is devoted to the determination of all the division normed real algebras satisfying some of the Moufang identities. In the case of the middle Moufang identity we prove that such an algebra $A$ is isomorphic to $R$, $C$, $H$ or $O$. For a division normed real algebra $A$ satisfying either the right or the left Moufang identity three more algebras occur in everyone of the cases.

1 DIVISION ONE-SIDED MOUANG ALGEBRAS

Theorem 1.1. Let $A$ be an algebra satisfying the right Moufang identity. Then $A$ is a division algebra if and only if there exists an alternative division algebra $B$ with same vector space as $A$ and product $\cdot$, and an involutive automorphism $f$ of $B$ such that the product $yx$ of any elements $y$, $x$ in $A$ is given in the following way:

$$yx = f(y)x$$

Proof. As it was previously pointed out, the existence of the division alternative algebra $B$ with the involutive automorphism $f$ implies that $A$ is a division algebra.

Assume $A$ is a division right Moufang algebra. Let $u \in A$, $u \neq 0$. Since $R_u$ is bijective, there exists a unique $e \in A$, $e \neq 0$ such that $eu = u$. For all $z \in A$ we have $e((uz)u) = ((eu)z)u = (uz)u$. The bijective character of $R_u \circ L_u$ implies $ey = y$ for all $y \in A$. Thus $e$ is a left unit in $A$. Since $R_e$ is a bijective map and $ye = y((ye)e) = ((ye)e)e$, then $y = (ye)e$. Moreover, $R_e$ is an automorphism of $A$. Indeed, for any $x$, $y \in A$ we have

$$R_e(yx) = (yx)e = ((ye)(xe)e = (ye)(xe)e$$

$$= (ye)(xe) = R_e(y)R_e(x).$$
Now we define
\[ y \cdot x = (ye)x \]
for any \( x, y \in A \). As it was previously pointed out, the new product makes the underlying vector space of \( A \) a right Moufang algebra \( B \). We assert that \( B \) is an alternative algebra and \( R_e \) is an involutive automorphism of \( B \). First we observe that \( B \) is a right alternative algebra. Indeed, if \( x, y \in A \) then we have
\[
y \cdot x^2 = y((xe)x) = (ye)((xe)x) = ((ye)x)e)x = ((ye)x)e = (y \cdot x)x.
\]
Taking into account the Kleinfeld theorem (see\(^{[6, \text{ p.} 345]}\)), \( B \) is an alternative algebra. On the other hand,
\[
R_e(x, y) = R_e((xe)y) = R_e(xe)R_e(y) = ((xe)e)(ye) = R_e(x)R_e(y)
\]
for any \( x, y \in A \). Moreover, \( R_e^2 = \text{Id} \). Therefore, the linear map \( R_e \) is an involutive automorphism of \( B \). We denote it by \( f \). Obviously \( B \) is a division algebra. Furthermore, for any \( x, y \in A \) we have \( yx = ((ye)e)x = (ye)x = f(y)x \).

Considering the opposite algebra we can obtain from Theorem 1.1. the following theorem.

**Theorem 1.2.** Let \( A \) be an algebra satisfying the left Moufang identity. Then \( A \) is a division algebra if and only if there exists an alternative division algebra \( B \) with same vector space as \( A \) and product \( \cdot \), and an involutive automorphism \( f \) of \( B \) such that the product \( xy \) of any elements \( x, y \) in \( A \) is given in the following way:
\[
xy = xxf(y).
\]

## 2 DIVISION NORMED REAL ALGEBRAS SATISFYING MOUFTANG IDENTITIES

The following theorem determines all the division normed real right Moufang algebras and it is an easy consequence of Theorem 1.1. (and its proof).

**Theorem 2.1.** Let \( A \) be a division normed real algebra which satisfies the right Moufang identity. Then \( A \) is isomorphic to one and only one of the following algebras:
1. One of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.
2. A two-dimensional algebra of basis $\{e, i\}$ and multiplication table
   $$e^2 = e, \quad ei = i = -ie$$
3. An algebra admitting a basis $\{e, i, j, k\}$ and multiplication table

$$
\begin{array}{cccc}
  e & i & j & k \\
  e & e & i & j & k \\
i & i & -e & k & -j \\
j & -j & k & e & -i \\
k & -k & -j & i & e \\
\end{array}
$$

4. The algebra of dimension 8, with base $\{u_1, \ldots, u_8\}$ and multiplication table

$$
\begin{array}{cccccccc}
  & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\
u_1 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\
u_2 & u_2 & -u_1 & u_4 & -u_3 & u_6 & -u_5 & -u_8 & u_7 \\
u_3 & u_3 & -u_4 & -u_1 & u_2 & u_7 & u_8 & -u_5 & -u_6 \\
u_4 & u_4 & u_3 & -u_2 & -u_1 & u_6 & -u_7 & u_8 & u_5 \\
u_5 & -u_5 & u_6 & u_7 & u_8 & u_1 & -u_2 & -u_3 & -u_4 \\
u_6 & u_6 & -u_5 & u_8 & -u_7 & u_2 & u_1 & u_4 & -u_3 \\
u_7 & -u_7 & -u_8 & -u_5 & u_6 & u_3 & -u_4 & u_1 & u_2 \\
u_8 & -u_8 & u_7 & -u_6 & -u_5 & u_4 & u_3 & -u_2 & u_1 \\
\end{array}
$$

**Proof.** By the proof of Theorem 1.1, $A$ is a division algebra if and only if there exist a left unit $e$ in $A$ such that $R_e^2 = \text{Id}$ and being alternative the algebra $B$ with the same vector space as $A$ and new product $\cdot$ defined by $y \cdot x = R_e(y)x$ for any $x, y \in A$. Also it was shown that $R_e$ is an involutive automorphism of the algebra $B$. The new norm $\|x\|' = \|e\|\|x\|$ makes $B$ a normed algebra. As it was previously pointed out, $B$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (see [5, Theorem 3.2]). It is well known that if $C = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ then all the automorphisms of $C$ are isometric maps and that every involutive automorphism $f$ of $C$ has two different possibilities: either, (i) $f = \text{Id}$; or (ii) the algebra $C$ is different from $\mathbb{R}$ and splits as a direct sum $\text{Sym}(C,f) \oplus \text{Sk}(C,f)$ where the symmetric elements $\text{Sym}(f)$ of $f$ are a real division algebra with dimension half of that of $C$. In the case that $R_e = \text{Id}$ the algebras $A$ and $B$ agree and the first possibility holds. By the previous comments, if $R_e \neq \text{Id}$ then we have the possibilities 2, 3, and 4, depending that $\dim A = 2, 4$ or 8.

In a similar way it is obtained the following theorem.
Theorem 2.2. Let $A$ be a division normed real algebra which satisfies the left Moufang identity. Then $A$ is isomorphic to one and only one of the following algebras:

1. One of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.
2. A two-dimensional algebra of basis $\{e, i\}$ and multiplication table
   \[ e^2 = e, \quad -ei = i = ie \]
3. An algebra admitting a basis $\{e, i, j, k\}$ and multiplication table

\[
\begin{array}{cccc}
  e & i & j & k \\
  e & e & i & -j & -k \\
  i & i & -e & k & j \\
  j & j & -k & e & -i \\
  k & k & j & i & e \\
\end{array}
\]

4. The algebra of dimension 8, with base $\{u_1, \ldots, u_8\}$ and multiplication table

\[
\begin{array}{cccccccc}
  u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\
  u_1 & u_2 & -u_1 & u_3 & -u_4 & u_5 & -u_6 & u_7 \\
  u_2 & -u_2 & u_1 & u_4 & u_3 & -u_5 & u_6 & -u_7 \\
  u_3 & -u_4 & -u_1 & u_2 & -u_3 & -u_5 & u_7 & -u_8 \\
  u_4 & u_3 & -u_2 & -u_1 & -u_5 & u_4 & u_5 & u_6 \\
  u_5 & u_5 & -u_6 & -u_7 & -u_8 & u_1 & u_2 & u_3 \\
  u_6 & u_7 & u_8 & u_5 & -u_6 & u_3 & u_4 & -u_2 \\
  u_7 & u_8 & u_7 & u_6 & u_5 & u_4 & u_3 & -u_2 \\
  u_8 & u_8 & -u_7 & u_6 & u_5 & u_4 & u_3 & -u_2 \\
\end{array}
\]

Theorem 2.3. Let $A$ be a division normed real algebra which satisfies the middle Moufang identity. Then $A$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Proof. Let $u \neq 0$ in $A$. Since $A$ is a division algebra, there exists a unique nonzero element $e \in A$ such that $u = eu$. So $(uy)u = (uy)(eu) = [u(ye)]u$ for all $y \in A$. Taking into account the bijective character of $R_u$ and $L_u$, we obtain $y = ye$. On the other hand, $e(ey) = (e(ey))e = (ee)(ye) = ey$. Thus $ey = y$ and $A$ has $e$ as a unit element. Making $y = e$ in Eq. (2) we obtain that $A$ is a flexible algebra. In particular, $x^2x = xx^2$ for all $x \in A$. On the other hand, as a consequence of (2) also it is obtained $(x^2)^2 = x^4$.

By\textsuperscript{[7]} Lemma 3, p.554, $A$ is a power-associative algebra. So for any $x \in A \neq 0$, the subalgebra $B_x$ generated by $e$ and $x$ is a unital associative commutative algebra. This algebra has not topological zero divisors. So by a well known Kaplansky's theorem\textsuperscript{[8]} Theorem 3.1 the algebra $B_x$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. In particular, $A$ is a quadratic algebra. The division flexible character of $A$
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jointly with the Osborn results\textsuperscript{[9], pp. 203–204} yields easily to the existence of an
involutive antiautomorphism \( s \) of \( A \) and a definite positive symmetric
bilinear form \( (\cdot, \cdot) \) such that \( s \) is an isometry map relative to \( (\cdot, \cdot) \) and being

\[
(xy|z) = (x|zs(y)) = (y|sxz)
\]

for any \( x, y, z \in A \) (see also\textsuperscript{[10]}). Let \( g: A \times A \to \mathbb{R} \) be the map defined by
\( g(x, y) = (x|s(y)) \) for all \( x, y \in A \). A straightforward verification shows that
\( g \) is an associative nondegenerate symmetric bilinear form. Now for any
\( x, y, z, u \in A \) we have

\[
g(y((xz)x) - ((yx)x)x, u) = g((xz)x, uy) - g(z, (xu)(yx))
\]

By the nondegenerate character of \( g \), the algebra \( A \) satisfies the right
Moufang identity. Since \( A \) is a unital algebra and the algebras of the possi-
bilities 2, 3, and 4 of Theorem 2.1 have not unit element, then we can assert
that \( A \) is isomorphic to \( \mathbb{R}, C, H \) or \( O \).

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