Communications in Algebra

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

COMPOSITION ALGEBRAS SATISFYING MOUFANG IDENTITIES

José Antonio Cuenca Mira

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga.
Málaga, 29080. Spain

First Published on: 12 April 2002
To cite this Article: José Antonio Cuenca Mira , 'COMPOSITION ALGEBRAS SATISFYING MOUFANG IDENTITIES', Communications in Algebra, 30:12, 5891 - 5899
To link to this article: DOI: 10.1081/AGB-120016020
URL: http://dx.doi.org/10.1081/AGB-120016020

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.informaworld.com/terms-and-conditions-of-access.pdf

This article maybe used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan or sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

© Taylor and Francis 2007
COMPOSITION ALGEBRAS SATISFYING MOUFANG IDENTITIES

José Antonio Cuenca Mira

Departamento de Álgebra, Geometría y Topología,
Facultad de Ciencias, Universidad de Málaga,
29080 Málaga, Spain
E-mail: cuenca@agt.cie.uma.es

ABSTRACT

This note is strongly inspired by [1] and [2] and it is devoted to the determination of all the composition algebras satisfying some of the Moufang identities.

Let $F$ be a field and $A \neq 0$ a nonassociative (i.e., not necessarily associative) $F$-algebra. Assume $n: A \to F$ is a nondegenerate quadratic form. The algebra $A$ is said to be a composition algebra if

$$n(xy) = n(x)n(y)$$

(1)

for any $x, y \in A$. Here the nondegenerate character of $n$ means that if $n(\cdot, \cdot)$ is the associate bilinear form given by $n(x, y) = n(x + y) - n(x) - n(y)$, then 0 is the only element $x$ in $A$ such that $n(x) = 0 = n(x, A)$. Obviously, if the characteristic of $F$ is different from 2 then $n(x) = \frac{1}{2}n(x, x)$, and so $n$ is nondegenerate iff 0 is the only element $x$ such that $n(x, A) = 0$. Notice that we do not impose the existence of a unit element in $A$. When this occurs the well known generalized Hurwitz Theorem asserts that $A$ is isomorphic to
one of the following algebras: (i) $F$; (ii) $F \oplus F$; (iii) a separable quadratic field extension of $F$; (iv) a generalized quaternion algebra; (v) a Cayley-Dickson algebra; (vi) a purely inseparable field extension of exponent one of the ground field $F$, being $F$ of characteristic 2 with $n(A, A) = 0$ and $x^2 = n(x)e$ for all $x \in A$. (see [3,4]). Each algebra of one of the types (i)–(v) will be called a Hurwitz algebra. In particular, if $F$ has characteristic $\neq 2$ and $A$ has unity then only Hurwitz algebras occur. Moreover, $A$ has dimension 1, 2, 4 or 8. However, if the existence of unity is dropped then can appear infinite dimensional algebras [5,6]. This is so even for one-sided division composition algebras with unity by the same side (see [7,8]). In the recent times have been given very complete description of composition algebras without unity but satisfying some additional condition as in everyone of the following cases:

1. associativiness of $n^{[9–14]}$;
2. flexible identity, ground field of characteristic $\neq 2$ and finite dimensionality $^{[15–17]}$;
3. characteristic $\neq 2, 3$, finite-dimensionality and associativiness of the third powers $^{[18,19]}$;
4. associativiness of third and forth powers $^{[2]}$ (see also $^{[20,21,18]}$);
5. degree two and ground field with more than five elements and characteristic $\neq 2, 3^{[19]}$.

We recall that an alternative algebra $A$ satisfies the following identities:

\begin{align*}
y((xz)x) &= ((yx)z)x \quad (2) \\
(xy)(zx) &= (x(yz))x \quad (3) \\
(x(yx))z &= x(y(xz)) \quad (4)
\end{align*}

These identities are called right, middle and left Moufang identity respectively.

In this note we determine all the composition algebras satisfying some of the Moufang identities. So in Sec. 2 we prove that every composition algebra which satisfies the middle Moufang identity has a unit element and therefore is isomorphic to a Hurwitz algebra when the ground field is of characteristic different from 2. On the other hand, for composition algebras satisfying a one-sided Moufang identity additional cases occur. We prove that everyone of these algebras is completely determined by a unital composition algebra $B$ and an involutive automorphism of $B$.

We recall that if $A$ is a composition algebra then from (1) it is easily obtained by linearization

\[ n(xy, xz) = n(x)n(y, z) = n(yx, zx) \quad (5) \]
and
\[ n(x, y)n(z, t) = n(zy, tx) + n(zx, ty) \]  
(6)
for any \( x, y, z, t \in A \). As a consequence of (5) we have that the linear maps of left and right multiplication \( L_x, R_x : A \rightarrow A \), given by \( L_x : y \mapsto xy \) and \( R_x : y \mapsto yx \), are injective for all \( x \in A \) such that \( n(x) \neq 0 \).

1 COMPOSITION ONE-SIDED MOUFANG ALGEBRAS

First we give some properties of the composition algebras satisfying the right Moufang identity.

Lemma 1.1. Let \( A \) be a composition algebra which satisfies the right Moufang identity. If \( x, t \in A \) then the following relations hold:

\[ n(x, (xt)x)^2 n(x^2 t) = n(x)n(x, (xt)x)n((x^2 t)x)t, x^2 t) \]
\[ = n(x)^3 n(t)n(x, (xt)x)n(x^2 t, x) \]
\[ = n(x)^3 n(t)n((x^2 t)x), x) + 2n(x)^3 n(t)^2 \]  
(7)

**Proof.** By (5) and (2) we have

\[ n(x, (xt)x)^2 n(x^2 t) = n(x, (xt)x)n(x, (xt)x)n(x^2 t) \]
\[ = n(x, (xt)x)n((x^2 t)x), (x^2 t)((xt)x)) \]
\[ = n(x)n(x, (xt)x)n(x^2 t, ((xt)x)x) \]

which proves the validity of the first equation of (7). The second one is an immediate consequence of (5). Now from (6), (5) and (2), we obtain

\[ n(x, (xt)x)n(x^2 t, x) = n((x^2 t)((xt)x), x^2 ) + n((x^2 t)x, x((xt)x)) \]
\[ = n(x)n((x^2 t)x)t, x) + n(x)n(x^2 t, x^2 t) \]

which completes the proof.

**Proposition 1.2.** Let \( A \) be a right Moufang composition algebra and \( x, t \in A \). Then the following assertions hold:

1. The relation
\[ n(x)^2 n(t)x + n(x)((x^2 t)x)t - n(x, (xt)x)x^2 t = 0 \]  
(8)
holds.
2. If \( n(t) \neq 0 \) then there exists a unique element \( e_{xt} \in A \) such that 
\[ e_{xt}t = x. \]
3. If \( n(x) \neq 0 \) then there exists a unique element \( e_x \) in \( A \) such that 
\[ e_x^2 = x. \]

On the other hand, taking into account Lemma 1.1 we obtain
\[ n(n(t)x + n(n((x^2t)x)t - n(x, (xt)x)x^2t, z) \]
\[ = n((x^2t)x, ((zx)t)x) + n(n(((x^2t)x)t)x, zx) \]
\[ - n((x^2t)x, (zx)t)x, n((x^2t)x)t, z)) - n((x^2t)((xt)x), zx) = 0 \] (9)

This jointly with (9) and the nondegenerate character of \( n \) gives (8).

2. Uniqueness of \( e_{xt} \) with the given property is a consequence of the injective character of \( R_t \). If \( n(x) \neq 0 \), making
\[ e_{xt} = \frac{1}{n(x)^2n(t)} (n(x, (xt)x)x^2 - n(x)(x^2t)x) \]
we obtain \( e_{xt}t = x \) because (8). It is well known that the nonsingular elements of a vector space \( V \) endowed with a nondegenerate symmetric bilinear
form span all \( V \). So \( e_x t = x \) remains true in the case of an arbitrary element \( x \) in \( A \).

3. is an immediate consequence from 2.

4. Newly uniqueness of \( x' \) is a consequence of the injectivity of \( R_x \).

Putting

\[
x' = \frac{1}{n(x)} \{ n(x, x^2) x - n(x) x^2 x \}
\]

we obtain \( x' x = e_x \). On the other hand, \( xx' = e_x \) is a consequence of the right Moufang identity. From \( e_x x = x, x' x = e_x \) jointly with (1) we obtain \( n(e_x) = 1 \) and \( n(x') \neq 0 \). So 3 and the previous reasoning guarantees the existence of elements \( e_x \) and \( (x')' \), determined in a unique way, such that 

\[
e_x x' = x' \quad \text{and} \quad (x')' x = e_x.
\]

Since \( (e_x x') x = ((x' x)x') x = x'(e_x x) = x' x \), we have \( e_x x' = x' \). Thus, \( e_x = e_x \) and \( (x')' = x \).

5. The right Moufang identity jointly with parts 2, 3 and 4 give 

\[
(x' x)t = x. \quad \text{Therefore,} \quad x't = e_x t.
\]

6. By linerization of the right Moufang identity it is obtained that

\[
u((v w)z) + u((z w)v) = ((u v)w)z + ((u z)w)v.
\]

This and the previous results give

\[
x + e_x e_x t = e_x ((x' t) t) + e_x ((t' i) x) = (e_x x') t + (e_x t') x \\
= (x' t) + ((e_x t') t') x = 2 x
\]

Therefore, \( e_x (e_x x) = x \). Taking into account that \( e_x x = x \), this implies \( e_x x = x \) and so \( e_t = e_x \).

7. Let \( e \) be the element of \( A \) which agrees with all the \( e_x \), \( n(x) \neq 0 \).

For all \( y \in A \) such that \( n(y) \neq 0 \) we have \( e y = e_y y = y \). Since any element in \( A \) is a linear combination of nonsingular elements, this holds for \( y \) arbitrary in \( A \). In particular \( e \) is an idempotent. So \( (y e) e = y (e^2 e) = y e \) for any \( y \in A \). Thus, \( R_e^2 = 1d \).

**Theorem 1.3.** Let \( A \neq 0 \) be an algebra that satisfies the right Moufang identity. Then \( A \) is a composition algebra if and only if there exists a unital composition algebra \( B \) of same vector space as \( A \), with product \( * \), and an involutive automorphism \( f \) of \( B \) such that the product \( y x \) of any elements \( y, x \) in \( A \) is given in the following way:

\[
y x = f(y) \cdot x.
\]
Proof. By Proposition 1.2, there exists a left unit $e$ such that $R^2_e = \text{Id}$. As in [4, p.25], the algebra $B$ of same vector space as $A$ but with product given by

$$y \cdot x = R_e(y)x$$

for any $x, y \in A$, is a composition algebra, which has $e$ as unity. Moreover, we have

$$R_e(x \cdot y) = ((xe)y)e = x((ey)e) = x(ye) = ((xe)(ye)\cdot R_e(x) \cdot R_e(y)$$

for any $x, y \in A$. So $R_e$ is an involutive automorphism of $B$. Moreover, for any $x, y \in A$ we have $yx = ((ye)x = (ye)x = R_e(y)x$. This completes the proof.

From Theorem 1.3 we obtain the following theorem considering the opposite algebra.

**Theorem 1.4.** Let $A \neq 0$ be an algebra that satisfies the left Moufang identity. Then $A$ is a composition algebra if and only if there exists a unital composition algebra $B$ of same vector space as $A$ and product $\cdot$ and an involutive automorphism $f$ of $B$ such that the product $xy$ of any elements in $A$ is given in the following way:

$$xy = x \cdot f(y).$$

## 2 COMPOSITION ALGEBRAS SATISFYING THE MIDDLE MOUFANG IDENTITY

We will need the following lemma.

**Lemma 2.1.** Let $A$ be a composition algebra which satisfies the middle Moufang identity. For every $x$ in $A$ the relation

$$n(x)^3 n(x, x^2)^2 = n(x)^4 n(x^2, x) + 2n(x)^6$$

holds.

**Proof.** Since $(xx^2)x^2 = (x(x^2))x$ and $x^2 x^2 = (xx^2)x$, by (5) and (6) we obtain
Proposition 2.2. Let $A$ be a composition algebra which satisfies the middle Moufang identity.

1. For any $x \in A$ the following relation holds:

$$n(x)x(x^2x) - n(x, x^2)x^2 + n(x)^2x = 0. \quad (11)$$

2. If $n(x) \neq 0$ then there exists a unique element $e_x$ such that $e_x x = x$.

Proof. 1. Since $(xx^2)x = x^2x^2$ and $(x(x^2x))x = (xx^2)x^2$, by (5) and (6) for any $y \in A$ we obtain

$$n(n(x)x(x^2x) - n(x, x^2)x^2 + n(x)^2x, y)$$

$$= n(x)n(x(x^2x), y) - n(x, x^2)n(xx^2, y) + n(x)^2n(x^2, y)$$

$$= n((x(x^2x))x, y) - n((xx^2)x^2, yx)$$

$$- n((xx^2)x, yx^2) + n(x^2x^2, yx^2) = 0 \quad (12)$$

On the other hand, from (5) and Lemma 2.1 we obtain

$$n(n(x)x(x^2x) - n(x, x^2)x^2 + n(x)^2x)$$

$$= n(x)^2n(x(x^2x)) + n(x, x^2)^2n(xx^2) + n(x)^4n(x)^2$$

$$- n(x)n(x, x^2)n(x(x^2x), x^2) + n(x)^3n(x(x^2x), x^2)$$

$$- n(x, x^2)n(x)^2n(xx^2, x^2)$$

$$= 2n(x)^6 - n(x)^3n(x, x^2)^2 + n(x)^4n(x^2, x, x) = 0 \quad (13)$$

Since $n$ is nondegenerate, from (12) and (13) we obtain (10).

2. If $n(x) \neq 0$ then the multiplication maps $L_x$ and $R_x$ are injective maps, and so from (13) it is obtained $n(x)x^2x - n(x, x^2)x^2 + n(x)^2x = 0$. Unicity of $e_x$ is a consequence of the injective character of $R_x$. If we define $e_x = \frac{1}{n(x)} \{ -n(x)x^2x + n(x, x^2)x \}$ then we get $e_x x = \frac{1}{n(x)} \{ -n(x)x^2x + n(x, x^2)x^2 \} = x$. 

$$n(x)^3n(x, x^2)^2 = n(x)^2n(x, x^2)n(xx^2, x^2)$$

$$= n(x)^2(n((xx^2)x^2, x^2x) + n((xx^2)x, x^2x))$$

$$= n(x)^3n(x(x^2x), x^2) + n(x)^4n(x^2, x^2)$$

$$= n(x)^4n(x^2, x) + n(x)^5n(x, x)$$

$$= n(x)^4n(x^2, x) + 2n(x)^6$$
Theorem 2.3. Let $A$ be a composition algebra which satisfies the middle Moufang identity. Then we have one of the following possibilities:

1. $A$ is isomorphic to a Hurwitz algebra;
2. The ground field $F$ has characteristic 2 and $A$ is isomorphic to a purely inseparable field extension of exponent one of $F$, with quadratic form $n$ satisfying $n(A,A) = 0$ and $x^2 = n(x)e$ for any $x \in A$.

Proof. Since $A$ agrees with the linear span of the nonsingular elements in $A$, we can choose an element $x$ in $A$ such that $n(x) \neq 0$. By the previous proposition, there is a unique element $e_x \in A$ such that $e_xx = x$. We assert that $e_x$ is a unit element in $A$. For to see this we argue in a similar way to [1, proof of Theorem 2.3]. For completeness we repeat here the cited reasoning. For $y \in A$ we have $(xy)x = (xy)(e_x x) = (x(ye_x))x$. Taking into account the injective character of $R_x$ and $L_x$, we obtain $y = ye_x$. On the other hand, $e_x(e_x y) = (e_x(e_x y))e_x = (e_x e_x)(ye_x) = e_x y$. Since $n(e_x) = 1$, the linear map $L_{e_x}$ is injective. Therefore, $e_x y = y$ and $e_x$ is a unit in $A$. Now the well known generalized Hurwitz Theorem concludes the proof.

ACKNOWLEDGMENTS

This work was partially supported by Ministerio de Ciencia y Tecnologa (PB97-1497 and BFM2001-1886), FEDER, AECI and PAI (FQM-0125).

REFERENCES


Received August 2001