Continuous Optimization

Phase I cycling under the most-obtuse-angle pivot rule

Pablo Guerrero-García *, Ángel Santos-Palomo

Department of Applied Mathematics, University of Málaga, 29071 Málaga, Spain

Received 24 February 2003; accepted 5 January 2004
Available online 17 June 2004

Abstract

It has been recently claimed that the most-obtuse-angle pivot rule is one of the best choices for Phase I linear programs based on the simplex method. In this short note we give two instances of Phase I cycling under such ratio-test-free rule, both when it is used to obtain primal feasibility and when trying to achieve dual feasibility with its unnormalized counterpart. A crash procedure that is not objective-driven might be the cause, and a non-simplex active-set generalization could be used instead.

© 2004 Elsevier B.V. All rights reserved.

Keywords: Linear programming; Most-obtuse-angle; Ratio-test-free; Phase I cycling

1. Introduction

In the celebrated primal (dual) simplex method for linear programming, there is only one min-ratio test, which is used to determine the variable to leave (enter) the basis. Although nowadays there is highly encouraging research in using two min-ratio tests to determine both the entering and leaving variable, there also have been proposals in the opposite direction, namely to use ratio-test-free rules to determine the leaving (entering) variable. This rules (closely related to the so-called most-obtuse-angle rules) replace the min-ratio-test with the selection of the greatest allowable denominator.

Pan has recently claimed [8] that the most-obtuse-angle pivot rule is one of the best choices for Phase I linear programs, supporting this claim with a computational study on the complete set of NETLIB problems [3] that do not have BOUNDS and RANGES sections in their MPS files. Although Pan does not rule out the possibility of cycling, he reports that “the safeguarding device against cycling turned out to be superfluous” and that he has “failed to construct an artificial program with which cycling will occur” and

---

* Corresponding author. Tel.: +34-952-137168; fax: +34-952-132766.
E-mail addresses: pablito@ctima.uma.es (P. Guerrero-García), santos@ctima.uma.es (Á. Santos-Palomo).
URL: http://www.satd.uma.es/matap/personal/pablito/.

0377-2217/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2003.06.048
he believes that “it is more difficult (if possible) to do so to the most-obtuse-angle rule than to conventional
rules”.

In this short note we give two instances of Phase I cycling under such ratio-test-free rule by applying the
Phase I method proposed by Pan (and its obvious dual counterpart) to the dual problems corresponding to
two classical examples of cycling given in an apparently different context. We also identify the cause and
sketch a possible remedy in Section 4. In order to highlight the underlying relationships, it turns out to be
beneficial in showing our results to use the following unsymmetric primal–dual pair of linear programs
using a non-standard notation (we have deliberately exchanged the usual roles of $b$ and $c$, $x$ and $y$, $n$ and $m$,
and $(P)$ and $(D)$, as e.g. in [4, §8]):

$$
(P) \min c^T x, \quad x \in \mathbb{R}^n
$$

$$
(D) \max b^T y, \quad y \in \mathbb{R}^m
$$

s.t. $A^T x \geq b$, \hspace{1cm} s.t. $Ay = c$, \hspace{1cm} $y \geq 0$,

where $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ and rank$(A) = n$. We denote with $\mathcal{F}$ and $\mathcal{G}$ the feasible region of $(P)$ and $(D)$, respectively. The notation used here is

$$
r_j = a^T_j x - b_j, \quad x = A^{-1}_B b_B, \quad \text{and} \quad y = A^{-1}_B c,
$$

where $a_j$ is the $j$th column of $A$; as usual, $B$ is the index set of basic variables and $N$ that of the non-basic
ones, and the current iteration is indicated by a superindex $(k)$ when it is not clear from context.

We assume that the reader is capable of applying to problem $(D)$ both the primal simplex algorithm
(starting from a vertex $y \in \mathcal{G}$ such that $|B| = n$), and the dual simplex algorithm (starting from a corre-
sponding vertex $x \in \mathcal{F}$) and then we shall not repeat it here. The primal feasibility cycling example given in
Section 2 occurs when applying to $(D)$ the primal simplex method with the most-obtuse-angle pivot row
rule to determine the leaving variable, whereas the dual feasibility cycling example given in Section 3 occurs
when applying also to $(D)$ the dual simplex method with the ratio-test-free pivot column rule to determine
the entering variable.

2. Primal feasibility cycling example

Pan’s proposal (cf. [8]) is equivalent to applying to $(D)$ the primal simplex method with the combination of
Dantzig’s column rule

$$
p = \arg \min \{ r_j < 0 : j \in N \}
$$

and the most-obtuse-angle row rule (also known as ratio-test-free row rule)

$$
q = B_l = \arg \max \{ \delta_{sl} : 0 : l \in 1 : n \}, \quad \delta_{sl} = A^{-1}_{sl} a_p,
$$

to obtain a primal feasible vertex with $|B| = n$, and then the dual simplex method to achieve optimality.
The starting point (not necessarily dual feasible) for the primal simplex method is obtained from an un-
detailed, not objective-driven, crash procedure. Unlike the use of a steepest-edge rule, Pan does not have to
compute several $\delta_{sl}$’s to decide the entering variable $p$, and hence the dependence of $\delta_{sl}$ on $p$ is not explicitly
shown.

The example given below is a minor adjustment of the dual program constructed from the feasibility
problem developed by Powell [13] to illustrate the cycling behaviour of the Phase I proposed by Rosen in
[14, pp. 210–211] for his gradient projection method. Rosen himself does not give any particular rule to
select an index $q$ from those for which $\delta_{sl} > 0$ hold, but Dax proposed in [2] precisely the most-obtuse-
angle row rule. Then Pan’s proposal cycles with:
\[(D_1) \quad \max -2[\gamma x, \beta, \gamma x, \beta; \gamma x, \beta, \gamma x, \beta, 0]y,\]
\[
\begin{bmatrix}
x & -\beta & 0 & -2\beta & -x & \beta & 0 & 2\beta & 0 \\
0 & -2\beta & -x & \beta & 0 & 2\beta & x & -\beta & 0 \\
-3x & 0 & -3x & 0 & -3x & 0 & -3x & 0 & 1
\end{bmatrix}y = c,
\]
\[y \in \mathbb{R}^9, y \geq 0,\]

where \(\mathcal{B}^{(0)} = \{8, 1, 9\}\) is taken as the (well-conditioned, quite sparse, quasi-triangular) initial basis that could be obtained by the crash procedure, \(\alpha = 10^{-1/2}, \beta = 5^{-1/2}\), and \(\gamma \in [-0.1, 0.1]\). The vector \(c = [x + 2\beta; -\beta; 1 - 3x]\) has been chosen as the sum of the columns indexed by \(\mathcal{B}^{(0)}\) for \(y^{(0)}\) to be dual feasible; although this is not mandatory for Pan’s method, it serves to illustrate (see the cyclical sequence of tableaux shown in Figs. 1 and 2) that dual feasibility is not maintained throughout this Phase I. Note that \(\|a_j\| = 1\) for all \(j \in 1:9\), and hence Pan’s proposal cycles even if the entering variable were selected according to Brown and Koopmans’ normalized column rule [1, p. 379]:

\[p = \arg \min \{r_j/\|a_j\| < 0 : j \in \mathcal{N}\}.\]

The notation used here for the tableau of the \(k\)th iteration \((k \in \mathbb{N})\) is

\[
\begin{array}{c|c|c|c}
\hline
\alpha & \beta & \beta & 0 & \beta & 0 \\
0 & -2\beta & -x & \beta & 0 & 2\beta & x & -\beta & 0 \\
-3x & 0 & -3x & 0 & -3x & 0 & -3x & 0 & 1 \\
\hline
\end{array}
\]

with \(m^* = m + 1\) and the entering (leaving) variable indicated by an up (right) arrow. Due to the symmetry revealed by the tableaux (already pointed out in [13]), it suffices to analyze iterations §2.0 and §2.1:

§2.0 Since all the residues \(r_j(j \in \mathcal{N})\) are linear functions of \(\gamma\), it is straightforward to establish that with

\[\gamma < \frac{\sqrt{2} - 1}{5\sqrt{2} - 3} \approx 0.1017,\]

we have

\[r_2 = 2\sqrt{2}\alpha(5\gamma - 1) < 2\alpha(3\gamma - 1) = r_3 < 0\]

and \(r_3 < r_j\) for all \(j \in \{4, 5, 6, 7\}\). Then \(r_2\) is the minimum residual and \(p = 2\) enters the basis according to Dantzig’s column rule. Furthermore, \(q = 8\) leaves the basis under the most-obtuse-angle row rule, as only \(\delta_8 > 0\).

§2.1 Again, due to the linearity of the residues with respect to \(\gamma\), it is easy to see that with

\[-\frac{1}{3} < \gamma < \frac{1}{2\sqrt{2} + 5} \approx 0.1277,\]

\(r_3 < 0\) and \(r_3 < r_2 < r_i\) hold for \(i \in \{4, 6, 7, 8\}\). Hence, \(p = 3\) enters the basis according to Dantzig’s column rule. Furthermore, \(q = 1\) leaves the basis if the most-obtuse-angle row rule is used, for \(\delta_1 = 1/2 > 1/(2\sqrt{2}) = \delta_2\).


\[ -2(\alpha + \beta) \quad 2\beta(5\gamma - 1) \quad 2\alpha(3\gamma - 1) \quad 4\beta \quad 4\alpha \gamma \quad 2\beta(3 - 5\gamma) \quad 2\alpha(1 - \gamma) \]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>1/√2</th>
<th>-1</th>
<th>0</th>
<th>-2</th>
<th>-1/√2</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-5√2</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>5√2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-15β</td>
<td>-9α</td>
<td>0</td>
<td>-6α</td>
<td>15β</td>
<td>3α</td>
<td>0</td>
</tr>
</tbody>
</table>

Iteration §2.0

\[ -\gamma(2\alpha + 5\beta) - \beta \quad \alpha(\gamma - 1) \quad \beta(5\gamma + 3) \quad 4\alpha \gamma \quad 4\beta \quad \alpha(3\gamma + 1) \quad \beta(1 - 5\gamma) \]

<table>
<thead>
<tr>
<th>1 + 5/√2</th>
<th>1/2</th>
<th>-5/√2</th>
<th>-1</th>
<th>0</th>
<th>-1/2</th>
<th>5/√2</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/(2√2)</td>
<td>-1/2</td>
<td>0</td>
<td>-1</td>
<td>-1/(2√2)</td>
<td>1/2</td>
<td>2</td>
</tr>
<tr>
<td>15β/2 + 1</td>
<td>-3α/2</td>
<td>-15β/2</td>
<td>-6α</td>
<td>0</td>
<td>-9α/2</td>
<td>15β/2</td>
<td>0</td>
</tr>
</tbody>
</table>

Iteration §2.1

\[ \alpha(2 - 4\gamma) + \beta(4 - 10\gamma) \quad 2\alpha(1 - \gamma) \quad 2\beta(5\gamma - 1) \quad 2\alpha(3\gamma - 1) \quad 4\beta \quad 4\alpha \gamma \quad 2\beta(3 - 5\gamma) \]

<table>
<thead>
<tr>
<th>-1/√2</th>
<th>-2</th>
<th>1/√2</th>
<th>-1</th>
<th>0</th>
<th>-2</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 5√2</td>
<td>2</td>
<td>-5√2</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>5√2</td>
<td>0</td>
</tr>
<tr>
<td>3α + 15β + 1</td>
<td>3α</td>
<td>-15β</td>
<td>-9α</td>
<td>0</td>
<td>-6α</td>
<td>15β</td>
<td>0</td>
</tr>
</tbody>
</table>

Iteration §2.2

\[ \alpha(\gamma + 1) + 2\beta \quad \alpha(3\gamma + 1) \quad \beta(1 - 5\gamma) \quad \alpha(\gamma - 1) \quad \beta(5\gamma + 3) \quad 4\alpha \gamma \quad 4\beta \]

<table>
<thead>
<tr>
<th>-1/2</th>
<th>-1/2</th>
<th>5/√2</th>
<th>1/2</th>
<th>-5/√2</th>
<th>-1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1/(2√2) - 1</td>
<td>-1/(2√2)</td>
<td>1/2</td>
<td>1/(2√2)</td>
<td>-1/2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-9α/2 + 1</td>
<td>-9α/2</td>
<td>15β/2</td>
<td>-3α/2</td>
<td>-15β/2</td>
<td>-6α</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Iteration §2.3

Fig. 1. Phase I cycling (first iterations) for Pan’s proposal.

Summing up, as shown in the tableaux, it suffices that

\[ \gamma \in [-0.1, 0.1] \subset \left( -\frac{1}{3}; \frac{\sqrt{2} - 1}{3\sqrt{2} - 3} \right) \]

for cycling to appear.
### 3. Dual feasibility cycling example

We could also pose the dual approach counterpart of Pan’s proposal \[12\], which consists of first starting the crash procedure and then applying to \((D)\) the dual simplex method with the combination of Dantzig’s row rule
\[
q = \beta_i = \arg \min \{ y_{B_i} : l \in 1 : n \}
\]
and the ratio-test-free column rule
\[
p = \arg \min \{ a_j^T d < 0 : j \in \mathcal{N} \}, \quad d = A_{\mathcal{N}}^{-T} e_i,
\]
to obtain a dual feasible vertex with $|\mathcal{B}| = n$, and then applying the primal simplex method to achieve optimality. (As in Section 2 with $\delta_d$ on $p$, the dependence of $d$ on $i$ is omitted due to similar reasons.) Note that the common way of writing the denominator of the dual simplex min-ratio test is as an entry in the tableau, rather than the inner product given above that occurs in its revised form. This is the reason why we have reserved the name *most-obtuse-angle column rule* for the normalized counterpart, namely

$$p = \arg\min \{a_j^T d / \|a_j\|_2 < 0 : j \in \mathcal{A} \}, \quad d = A_{\mathcal{A}}^T e_i.$$ 

Let us illustrate the danger of using the ratio-test-free column rule during Phase I. In this case the Phase I cycling example is the dual of a linear program that Gill, Murray and Wright attribute to Kuhn in [4, p. 351]:

$$(D_2) \quad \max \quad O^T y,$$

s.t. 

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 2 & -1/3 \\
0 & 1 & 0 & 0 & 9 & -1 \\
0 & 0 & 1 & 0 & -1 & 1/3 \\
0 & 0 & 0 & 1 & -9 & 2
\end{bmatrix} y = \begin{bmatrix}
-2 \\
-3 \\
1 \\
12
\end{bmatrix},$$

$y \in \mathbb{R}^5, y \geq O.$

\begin{tabular}{cccc|c|c|c}
0 & 0 & 0 & 7 & 0 & 0 & 7 \\
\hline
-2 & 2 & -1/3 & 1 & -1 & -1/3 & -1 \\
-3 & 9 & -1 & 2 & 0 & 1/3 & 2 \\
1 & -1 & 1/3 & 5 & 6 & 2 & 9 \\
12 & -9 & 2 & 4 & 3 & -1 & -9 \\
\hline
\end{tabular}

\begin{tabular}{cccc|c|c|c}
0 & 2 & 3 & N \backslash B & 0 & 2 & 3 & N \backslash B \\
\hline
\text{Iteration } 3.0 & \text{Iteration } 3.1 \\
0 & 0 & 0 & 7 & 0 & 0 & 7 \\
\hline
-2 & 2 & -1/3 & 1 & -1 & -1/3 & -1 \\
-3 & 9 & -1 & 2 & 0 & 1/3 & 2 \\
12 & -9 & 2 & 4 & 3 & -1 & -9 \\
1 & -1 & 1/3 & 5 & 6 & 2 & 9 \\
\hline
\end{tabular}

\begin{tabular}{cccc|c|c|c}
0 & 1 & 2 & N \backslash B & 0 & 1 & 2 & N \backslash B \\
\hline
\text{Iteration } 3.2 & \text{Iteration } 3.3 \\
0 & 0 & 0 & 7 & 0 & 0 & 7 \\
\hline
-3 & 9 & -1 & 2 & -1 & -1/3 & -1 \\
-2 & 2 & -1/3 & 1 & 6 & 9 & 2 \\
12 & -9 & 2 & 1/3 & 0 & 2 & -1 \\
1 & -1 & 1/3 & 5 & 3 & -9 & -1 \\
\hline
\end{tabular}

\begin{tabular}{cccc|c|c|c}
0 & 2 & 3 & N \backslash B & 0 & 2 & 3 & N \backslash B \\
\hline
\text{Iteration } 3.4 & \text{Iteration } 3.5 \\
0 & 1 & 2 & N \backslash B & 0 & 1 & 2 & N \backslash B \\
\hline
\end{tabular}

Fig. 3. Phase I cycling for dual counterpart of Pan’s proposal.
The (perfectly conditioned, diagonal) initial basis is $\mathcal{B}^{(0)} = \{1, 2, 3, 4\}$, and hence $y^{(0)} = [-2; -3; 1; 12; 0; 0]$ is not dual feasible but its corresponding vertex $x = O$ is primal feasible (although this would not either be mandatory in this dual counterpart of Pan’s proposal, as in [7]); the method proceeds cyclically as shown in Fig. 3, and as above, we only have to analyze iterations §3.0 and §3.1:

(§3.0) According to Dantzig’s row rule, $q = 2$ is the leaving variable ($y_2 = -3 < -2 = y_1$); since $a_2^T d = 9 > 0$ and $a_5^T d = -1 < 0$, $p = 2$ is the entering variable if the ratio-test-free column rule is used.

(§3.1) After selecting $q = 1$ to leave the basis (only $y_1 < 0$), $a_2^T d = -1 < -1/3 = a_1^T d$ and hence $p = 5$ is the entering variable.

In this example, the use of the most-obtuse-angle column rule instead of the ratio-test-free column rule avoids the cycle, but we think that a Phase I cycling example with $\|a_j\|_2 = 1$ can also be constructed.

4. Concluding remarks

In this short note we have given two Phase I cycling examples under the most-obtuse-angle pivot rule; the fact that the crash procedure is not objective-driven might be the cause. The first one serves to underline the connections with Rosen’s 1960 gradient projection method (and hence with the dual simplex method) when applied to linear programming, the Phase I algorithm he proposed and its celebrated Powell’s cycling counterexample 20 years later. The second cycling example serves to highlight the relation between Gill, Murray and Wright’s 1991 work and Pan’s dual approach counterpart. We hope this has shed some light over the subtleties of the proposed computational framework.

It is worth noting that we have also coped with non-simplex active-set methods (also currently known as basis-deficiency-allowing simplex variations [9–11]), namely those that allow non-square working-set matrices with $|\mathcal{B}| \leq n$. The main advantage in the context of this paper is that the crash procedure can be dispensed with: rather than using a crash procedure and a Phase I before the Phase II, we can directly use a Phase I alone. Therefore, this generalization to non-simplex active-set methods avoids the disastrous use (only from the cycling viewpoint, of course) of a crash procedure that is not objective-driven.

Several distinct Phase I’s are applicable (cf. [6,15]), all of them starting from scratch and taking the objective function into account. In particular, we have not been able to construct any Phase I cycling example when applying a primal non-simplex active-set method with the most-obtuse-angle row rule to a Gass’ single-artificial-variable Phase I; quite the contrary, we have obtained encouraging results [5, §5] with a sparse implementation.

Acknowledgements

The authors gratefully acknowledge the editor and the referees for their helpful corrections and remarks, which have improved the quality of the paper. We also thank one referee for providing us the references [7,12] that we were not aware of.

References