Piecelinearized methods for initial-value problems
with oscillating solutions

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Abstract

Two piecelinearized methods for the numerical solution of nonlinear second-order differential equations which contain the first-order derivative are presented. These methods provide piecelinear analytical solutions in open intervals, result in explicit finite difference equations for both the displacement and the velocity or only the displacement, at nodal points, and continuous solutions everywhere. One of the piecelinearized methods is a two-level method that provides smooth solutions and is self-starting; the other is a non-self-starting three-level technique. The paper also presents several iterative schemes, a time-piecelinear three-point technique and a time-piecelinear Numerov method. An extensive assessment of the piecelinearized methods is carried out in order to assess their accuracy in nine examples, and it is shown that, in some cases, these techniques provide as accurate results as exponentially- or trigonometrically-fitting methods that do require an estimate of the dominant frequency of the solution. The piecelinearized methods presented here do not require such an estimate at all, but may exhibit spurious fixed points/attractors in nonlinear problems and their accuracy is a strong function of both the nonlinearities and the time step. However, they are applicable to nonlinear second-order differential equations which contain the first-order derivative.

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Keywords: Second-order ordinary differential equations; Periodic solutions; Piecelinearized methods; Nonlinear systems

1. Introduction

Interest in the solution of second-order, ordinary differential equations whose solutions exhibit oscillatory or periodic behavior has arisen in several fields such as, for example, celestial mechanics, nonlinear dynamics, electronics, acoustics, molecular dynamics, quantum mechanics, biochemical systems such as, for example, rhythmic processes, cell cycles and circadian rhythms where the oscillations arise from biochemical reactions [1], etc.

Nonlinear oscillators with periodic solutions appear frequently in mechanical systems such as joints, gear trains, and linkages. Typical nonlinearities may be caused by wear or manufacturing tolerance, and Hertzian contact friction, to name but a few [2–4]. For linear periodic systems, there is a well-developed theory due to
Liapunov and Floquet that allows to calculate the characteristics exponents, the analytical solution and the stability of systems of linear, first-order, ordinary differential equations with periodic coefficients [5,6]. In the physics literature, the analysis of such systems is usually associated with Bloch.

For nonlinear systems with periodic solutions, there is a large variety of asymptotic methods based on the presence of a small parameter that allow to determine the (approximate) asymptotic solution of the periodic motion in powers of the small perturbation parameter [7–9], such as, for example, the harmonic balance method, the Krylov–Bogoliubov–Mitropolski (KBM) technique, averaging procedures, the method of multiple (time) scales, etc.

When the governing equations of nonlinear oscillators do not contain a small perturbation parameter, the analysis of these oscillators must be carried out numerically, unless the governing equations are piecewise linearized and analytical solutions to these piecewise linear equations are easily obtained. For example, the author has recently proposed a piecewise linearized method for the analysis of nonlinear oscillators with limit cycles that is based on the linearization of single degree-of-freedom problems, results in a nonlinear map at discrete times [10,11] and provides piecewise analytical solutions which are smooth everywhere by imposing continuity of the displacement and velocity at the end points of two-point intervals. Although this technique has been shown to be very accurate for some single degree-of-freedom problems, its extension to multiple degree-of-freedom systems is rather complex, for it requires that either the Schur normal form theorem or the Jordan canonical form of a certain matrix be obtained at each time step, or a matrix exponential be approximated by means of, for example, a Padé approximation or Taylor’s series expansions.

Other approximate methods for the determination of periodic orbits include the one developed by Starkov [12] who considered the localization of periodic orbits of polynomial vector fields of even degree by means of linear functions based on the solution of conditional extremum problems. Tadi [13] used an iterative procedure based on a variational principle to determine points of the periodic orbit. Both Starkov and Tadi considered systems of nonlinear, first-order ordinary differential equations.

There is now a vast variety of methods for an efficient integration of second-order ordinary differential equations with oscillatory solutions such as, for example, P-stable methods, techniques that minimize the phase lag, exponential-fitting algorithms, Scheifele G-functions-based schemes, etc. Some of these numerical techniques require a good estimate of either the period or the dominant frequency of the solution, and the coefficients that appear in these techniques depend on \( \theta = \omega k \), where \( \omega \) is the estimated frequency and \( k \) is the time step. In addition, most of these techniques have been developed for either first-order ordinary differential equations or second-order differential equations which do not include the first-order derivative. The methods presented in this paper, however, have been developed for second-order ordinary differential equations that they do include the first-order derivative or velocity.

Some of the available numerical methods for initial-value problems with oscillating solutions that consider systems of first-order, ordinary differential equations include explicit Runge–Kutta schemes of high algebraic order [14,15] or minimal phase lag [16], Numerov’s method which has an interval of periodicity equal to (0, 6) and a phase-lag of order 4, explicit Numerov’s methods with reduced number of stages [17], linear multistep techniques which require a good estimate of the dominant frequency, exponentially-fitted Runge–Kutta algorithms without [18,19] or with embedding [20,21], explicit and implicit exponentially-fitted backward-difference algorithms [22], trigonometrically-fitted predictor–corrector schemes based on Adams–Bashforth formulae for the predictor and Adams–Moulton formulae for the corrector [23,24], etc.

Exponentially-fitted methods use a basis of functions that includes polynomials, exponential- and trigonometric-type solutions, and integrate exactly appropriate trigonometric polynomials of a given degree, just as classical finite difference methods integrate exactly algebraic polynomials. For problems with periodic or oscillatory solutions, Gautschi [25] was the first to propose a trigonometrically-fitted method which required that the frequency be estimated in advance. Since then, many exponentially-fitted linear multistep techniques have been developed especially for second-order ordinary differential equations where the first-order derivative is absent. Also, for first-order differential equations, especially tuned methods have been developed. However, all these exponentially-fitted techniques pose the problem of how to choose the frequencies. This problem has recently been solved for exponentially-fitted multistep methods where the frequencies are tuned to be as optimal as possible for systems of first-order ordinary differential equations [26–29]. By means of such an optimization procedure, an order of accuracy may be gained with respect to the underlying classical
multistep method, but such an optimization also shows that the optimal frequencies do not reflect the solution itself, but the behavior of some higher-order derivatives of it [30]. This optimization procedure has been recently extended for the frequency evaluation of exponentially-fitted RadauIIA methods [19]. The methods presented in this paper, however, do not require an evaluation or estimate of the frequency because they are based on the piecewise linearization of the second-order, ordinary differential equations and provide piecewise exponential/trigonometric solutions whose frequency is related to the Jacobian of the nonlinearities.

Some of the available methods for initial-value problems with periodic solutions that deal directly with second-order ordinary differential equations include exponentially-fitted, trigonometric-fitted [31–36] and phase-fitted techniques [14,15], hybrid multistep explicit Numerov techniques based on Taylor’s series expansions at the internal nodes [37,38], hybrid multistep explicit algorithms [39], symmetric multistep methods including Obrechkoff schemes [40–44], exponentially-fitted and trigonometrically-fitted Runge–Kutta–Nyström algorithms [45,46], adiabatic midpoint rule and adiabatic Magnus techniques based on the transformation of the problem to adiabatic variables and expansion of the oscillatory integrals [47], modified Scheifele G-function algorithms based on Taylor’s series expansion that require a previous knowledge of a good approximation to the frequency [48], linear multistep techniques, (implicit) predictor–corrector schemes, explicit algorithms, Bessel and Neumann series-based techniques, algebraic order methods, etc. [15].

It must be pointed out that exponentially-fitted methods integrate exactly a conveniently chosen mixture of monomials and exponential/trigonometric functions [29] and have been developed for second-order ordinary differential equations that do not contain first-order derivatives. Examples of exponentially-fitted (EF) methods include the exponentially-fitted Runge–Kutta (EFRK) and Runge–Kutta–Nyström (EFRKN) schemes which have been developed for the integration of \( y' = f(t,y) \) and \( y'' = f(t,y) \), respectively. As indicated previously, exponentially- or trigonometrically-fitted methods require an estimate of the dominant frequency [26], whereas the piecewise linearization methods presented in this paper do not.

Phase-fitted methods were introduced by Raptis and Simos [49] and are based on the analytical solution of linear problems having a periodic or oscillatory solution. These techniques include the Scheifele G-functions-based schemes which are based on refining the classical method of power series or B-series [50], algebraic order algorithms, explicit and implicit schemes, etc.

In order to analyze the stability and phase lag of numerical methods for the numerical integration of

\[
y'' = f(t,y), \quad y(0) = y_0, \quad y'(0) = y'_0,
\]

Lambert and Watson [51] introduced the following scalar equation:

\[
y'' = -\omega^2 y,
\]

where \( \omega \) is a real constant, and the interval of periodicity. When a two-step method is used to solve Eq. (2), one finds the following characteristic equation:

\[
\lambda^2 - Q(h^2)\lambda + R(h^2) = 0,
\]

where \( h = \omega k \). The dispersion and dissipation errors are

\[
\phi(h) = h - \arccos \left( \frac{Q(h^2)}{2\sqrt{R(h^2)}} \right), \quad d(h^2) = 1 - \sqrt{R(h^2)},
\]

respectively, and a method is said to be dispersive of order \( q \) and dissipative of order \( r \) if

\[
\phi(h) = O(h^{q+1}), \quad d(h) = O(h^{r+1}),
\]

respectively.

The numerical method is unconditionally stable provided that \( |\lambda_i| \leq 1 \) for all values of \( h^2 \), where \( \lambda_i \) with \( i = 1, 2 \) are the roots of Eq. (3). In addition, it is said that the numerical method has an interval of periodicity \((0, h_0^2)\), if, for all \( h^2 \in (0, h_0^2) \) and \( R(h^2) = 1 \), the roots of the characteristic Eq. (3) are \( \lambda_1 = \exp(i\theta h^2)) \) and \( \lambda_2 = \exp(-i\theta h^2)) \), where \( \theta(h^2) \) is a real function of \( h^2 \), i.e., \( |Q(h^2)| < 2 \). The method is P-stable, if the interval of periodicity is \((0, \infty)\). Methods that satisfy \( R(h^2) = 1 \) and \( |Q(h^2)| < 2 \) are called zero-dissipative, i.e., \( d(h) = 0 \).
On the other hand, when a method has a finite order of dissipation, the integration process is stable if the coefficients of the characteristic polynomial satisfy the following conditions:

$$|R(h^2)| < 1, \quad |Q(h^2)| < 1 + R(h^2)$$

(6)

for all $h \in (0, h_0)$, where $(0, h_0)$ denotes the interval of absolute stability of the method.

In this paper, we present two piecewise-linearized methods for the determination of the approximate periodic solution the following equation:

$$\ddot{y} = f(t, y, \dot{y}), \quad t > 0,$$

(7)

where $y \in \mathbb{R}^N$ and $f \in \mathbb{R}^N$, subject to $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$, and the dot denotes differentiation with respect to $t$. The methods are based on piecewise linearization, involve two or three time levels and provide piecewise continuous solutions. In the case of two time levels, the piecewise linearization technique presented here provides piecewise analytical solutions which are smooth everywhere, and explicit finite difference equations for both the displacement and the velocity at the nodal points, which are self-starting [10,11]. In the case of three time levels, the piecewise linearization technique provides piecewise analytical solutions which are continuous everywhere, and explicit finite difference equations for the displacement which are not self-starting.

The paper has been arranged as follows. In Section 2, piecewise linearization methods that provide smooth and continuous solutions everywhere are presented for $N = 1$. In that section, the stability of these techniques is also analyzed. In Section 3, the generalization of the piecewise-linearization methods presented in this paper to multi-degree of freedom systems, i.e., $N > 1$, is performed, while, in Section 4, an extensive assessment of the piecewise-linearized methods presented in this paper is presented as a function of the time step for several second-order ordinary differential equations. Such an assessment is carried out by comparing the numerical solutions with analytical or approximate ones, when available, and with the results of the second- and fourth-order accurate Störmer–Cowell–Verlet and Numerov methods, respectively, which are frequently used in molecular dynamics simulations. In addition, linearized-implicit finite difference techniques are also presented in Section 4. The paper ends with a brief summary of the main conclusions.

2. Piecewise-linearized methods for single degree-of-freedom problems

In this section, we consider a scalar equation of the form

$$\ddot{y} = f(t, y, \dot{y}), \quad t > 0,$$

(8)

i.e., $N = 1$ in Eq. (7), subject to $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$, where $f$ is continuous and differentiable with respect to $t$, $y$ and $\dot{y}$. Eq. (8) contains $\dot{y}$ in its right-hand side.

2.1. Piecewise-linearized smooth method

Owing to the nonlinearity of $f$, Eq. (8) cannot be solved, in general, in closed form and, therefore, approximate solution techniques have to be developed. Here, we consider a time interval $T$ so that $0 \leq t \leq T$ and divide it into subintervals $[t_n, t_{n+1}]$ such that the union of these subintervals is $[0, T]$ and $t_0 = 0$.

If, in each interval $(t_n, t_{n+1})$, Eq. (8) is linearized with respect to the previous time level, one can easily obtain the following second-order, linear, ordinary differential equation [10]:

$$\ddot{y} = f_n + T_n(t - t_n) + J_n(y - y_n) + H_n(\dot{y} - \dot{y}_n), \quad t_n < t < t_{n+1},$$

(9)

provided that $T_n$, $J_n$ and $H_n$ are defined at $t_n$, where $f_n \equiv f(t_n, y_n, \dot{y}_n)$, $T_n \equiv \frac{df}{dt}(t_n, y_n, \dot{y}_n)$, $J_n \equiv \frac{df}{dy}(t_n, y_n, \dot{y}_n)$ and $H_n \equiv \frac{df}{d\dot{y}}(t_n, y_n, \dot{y}_n)$, and the right-hand side of Eq. (8) has been linearized with respect to its three arguments. Eq. (9) is subject to $y(t_n) = y_n$ and $\dot{y}(t_n) = \dot{y}_n$.

If $J_n$ is not defined, then Eq. (8) is only linearized with respect to $t$ and $\dot{y}$ provided that $H_n$ is defined and, in this case, Eq. (9) applies without the third term in its right-hand side. Analogously, if $H_n$ is not defined, then Eq. (8) is only linearized with respect to $t$ and $y$ provided that $J_n$ is defined and, in this case, Eq. (9) applies without the fourth term in its right-hand side. If both $J_n$ and $H_n$ are not defined, then Eq. (8) is only linearized with respect to $t$ and, in this case, Eq. (9) applies without the third and fourth terms in its right-hand side.
Since the three possibilities just described are all included in Eq. (9), i.e., they correspond to setting $H_n = 0$, $J_n = 0$ and $J_n = H_n = 0$, respectively, in that equation, hereon we shall only consider Eq. (9).

Eq. (9) can be solved analytically in each interval $t_n \leq t < t_{n+1}$ and the form of its solution depends on the roots of the characteristic equation $\lambda^2 - H_n \lambda - J_n = 0$ and on the values of $H_n$ and $J_n$. For example, if $\frac{H_n^2}{4} + J_n < 0$ and $J_n \neq 0$, then the solution to Eq. (9) can be written as

$$y(t) = \exp(\mu_n(t-t_n))\left(A_n \cos(\Omega_n(t-t_n)) + B_n \sin(\Omega_n(t-t_n))\right) + \alpha_n + \beta_n(t-t_n),$$

(10)

where $\mu_n = \frac{1}{2}H_n$, $\Omega_n = -(\frac{H_n^2}{4} + J_n)$, $\beta_n = -\frac{t_n}{J_n}$, $\alpha_n = -\frac{H_n}{J_n} \beta_n - \frac{t_n}{J_n}$, $P_n = f_n - J_n y_n - H_n \dot{y}_n$, and the values of $A_n$ and $B_n$ can be easily determined from the conditions $y(t_n) = y_n$ and $\dot{y}(t_n) = \dot{y}_n$ as

$$A_n = y_n - \alpha_n,$$

(11)

$$B_n = -\frac{1}{\Omega_n}(\dot{y}_n - A_n \mu_n - \beta_n).$$

(12)

Eq. (10) yields the following explicit finite difference expressions for the conditions on the discriminant and $J_n$ considered above:

$$y_{n+1} = y(t_{n+1}) = \exp(\mu_n \Delta_n)(A_n \cos(\Omega_n \Delta_n) + B_n \sin(\Omega_n \Delta_n)) + \alpha_n + \beta_n \Delta_n,$$

(13)

$$\dot{y}_{n+1} = \dot{y}(t_{n+1}) = \exp(\mu_n \Delta_n)(A_n \dot{\Omega_n} \Delta_n + B_n \Omega_n \Delta_n) + (B_n \mu_n - A_n \Omega_n \sin(\Omega_n \Delta_n)) + \beta_n,$$

(14)

where $\Delta_n = t_{n+1} - t_n$ is the (possibly variable) time step.

Analytical solutions to Eq. (9) for other values of the discriminant, $J_n$ and $H_n$ can be found in any good textbook on linear ordinary differential equations, e.g. [52], and are not reported here.

The piecewise analytical method described above is self-starting and provides the values of the displacement and velocity in an analytical fashion. In addition, the method provides analytical solutions in the interval $(t_n, t_{n+1})$ which are smooth everywhere on account of the continuity of $y$ and $\dot{y}$ at $t_n$. Moreover, the method does not require an estimate of the frequency of oscillation and is valid for second-order ordinary differential equations which include the first-order derivative. When this derivative is absent, i.e., $f(t, y, \dot{y}) = f(t, y)$ and, therefore, $H_n = 0$, then $\mu_n = 0$ and Eqs. (13) and (14) become

$$y_{n+1} = A_n \cos(\Omega_n \Delta_n) + B_n \sin(\Omega_n \Delta_n) + \alpha_n + \beta_n \Delta_n,$$

(15)

$$\dot{y}_{n+1} = B_n \Omega_n \cos(\Omega_n \Delta_n) - A_n \Omega_n \sin(\Omega_n \Delta_n)) + \beta_n,$$

(16)

where

$$A_n = y_n - \alpha_n,$$

(17)

$$B_n = -\frac{1}{\Omega_n}(\dot{y}_n - \beta_n).$$

(18)

For the scalar equation

$$\ddot{y} = -\omega^2 y,$$

(19)

i.e., $f(t, y, \dot{y}) = -\omega^2 y$ where $\omega$ is a real number, it is an easy exercise to show that the method presented in this section provides the exact solution of Eq. (8) in exact arithmetic.

It must be pointed out that the piecewise-linearized smooth method presented here is of the exponential/trigonometric type and is based on the piecewise linearization of Eq. (8) about the previous time level and the fact that $t$, $y$ and $\dot{y}$ in $f(t, y, \dot{y})$ are considered as three different independent variables. Under this assumption, one can also expand the right-hand side of Eq. (8) with respect to $(t', y, \dot{y})$ where $t' \in [t_n, t_{n+1}]$ and keep the constant and linear terms of this expansion, so that, the resulting second-order linear ordinary differential equation can be integrated analytically in $(t_n, t_{n+1})$ subject to the same initial conditions as above. The case corresponding to $t' = t_n$ results in Eqs. (13) and (14), and similar expressions can be obtained for other values of $t'$, which are not reported here.

For the scalar test equation $\ddot{y} = -\omega^2 y$ where $\omega \in \mathbb{R}$ is constant, it is an easy exercise to show that Eq. (9) yields
which can be written in matrix form as \( y_{n+1} = Ay_n \) where \( y_n = (y_n, \dot{y}_n)^T \), the superscript T denotes transpose and

\[
A = \begin{pmatrix}
  \cos(\omega k) & \frac{1}{\omega} \sin(\omega k) \\
  -\omega \sin(\omega k) & \cos(\omega k)
\end{pmatrix}.
\]

The eigenvalues of \( A \) are \( \lambda_2 = \bar{\lambda}_2 = \cos(\omega k) + i \sin(\omega k) \), i.e., \( |\lambda_i| = 1 \), and the two eigenvalues are complex conjugate. Furthermore, if \( \omega k \neq m\pi \), where \( m \) is a natural number including zero, then it is easy to show that the eigenvectors associated with \( \lambda_1 \) and \( \bar{\lambda}_1 \) are \( (1, i\omega) \) and \( (1, -i\omega) \), respectively, and \( A \) can be diagonalized and written in Jordan’s canonical form as \( z_{n+1} = Az_n \) where \( z_n = P^{-1}y_n \), \( A = P^{-1}AP \) where \( A \) is a diagonal matrix of components equal to \( \lambda_1 \) and \( \bar{\lambda}_1 \). Therefore, \( z_{n+1} = \tilde{\lambda}_1 z_n \) and \( \tilde{\lambda}_1 \tilde{z}_n \), with \( |z_{n+1}| = |z_n| \) and \( |\tilde{z}_{n+1}| = |\tilde{z}_n| \), so that the piecewise-linearized method presented in this paper is stable. If \( \omega k = m\pi \), then \( A = I \) for \( m \) even, and \( A = -I \) for \( m \) odd, where \( I \) denotes the \( 2 \times 2 \) unit matrix; in both cases the eigenvalues of \( A \) have a magnitude equal to unity and are positive for \( m \) even, and negative, for \( m \) odd.

### 2.2. Piecewise-linearized continuous method

In the interval \([t_{n-1}, t_{n+1}]\), the right-hand side of Eq. (8) can be approximated by its Taylor’s series expansion about \( t_n \), and Eq. (8) can be, therefore, be approximated by Eq. (9) whose solution is Eq. (10) [11]. However, the conditions required for the determination of \( A_n \) and \( B_n \) are now \( y(t_{n-1}) = y_{n-1}, \quad y(t_n) = y_n, \) and \( y(t_{n+1}) = y_{n+1} \), which yield, for \( \frac{\nu^2}{4} + J_n < 0 \) and \( J_n \neq 0 \),

\[
A_n = y_n - x_n, \quad \text{and} \quad B_n = \frac{1}{\sin(\Omega_n k)} \left((y_{n-1} - x_n + \beta_n k) \exp(\mu_n k) + (y_{n+1} - x_n - \beta_n k) \exp(-\mu_n k)\right),
\]

and the three-point finite difference equation

\[
y_{n-1} \exp(\mu_n k) - 2 \cos(\Omega_n k)y_n + y_{n+1} \exp(-\mu_n k) = 2x_n(\cosh(\mu_n k) - \cos(\Omega_n k)) - 2\beta_n k \sinh(\mu_n k),
\]

where the time step has been assumed to be constant. Eq. (25) is not self-starting.

In order to provide starting values at \( n = 1 \), we shall use the initial condition on \( y \) and determine \( y_1 \) as \( y_1 = y_0 + \dot{y}_0 k + \frac{k^2}{2} f(0, y_0, \dot{y}_0) + O(k^3) \) which comes from the Taylor series expansion of \( y(k) \) around \( y(0) \) and the use of Eq. (8).

For \( f(t, y, \dot{y}) = f(t, y) \) and \( J_n < 0 \), Eq. (25) can be written as

\[
y_{n+1} - 2y_n + y_{n-1} = 2 \frac{f_n}{\Omega_n^2} \left(\cos \left(\sqrt{\Omega_n} k\right) - 1\right),
\]

where \( \Omega_n = \sqrt{-J_n} \). If \( \Omega_n k \ll 1 \), a Taylor series expansion of the cosine function in Eq. (26) yields

\[
y_{n+1} - 2y_n + y_{n-1} = f_n k^2 + O(k^4),
\]

which is the finite difference equation of the second-order accurate Störmer–Cowell–Verlet method. Therefore, the formulation presented here, cf., Eq. (25), yields as a special case a well-known method. It should be noted that the standard derivation of the Störmer method is based on the expansion of \( y(t_n + k) \) and \( y(t_n - k) \) about \( y(t_n) \) and the use of Eq. (8). Moreover, the formulation presented here implies that the Störmer method is applicable whenever \((-J_n)k^2\) is much smaller than one.

Since Eq. (25) was derived under linearization by imposing continuity conditions at three successive time levels, the piecewise-linearized continuous method presented here provides piecewise analytical solutions in \((t_{n-1}, t_{n+1})\) which are continuous in \([0, T]\). Moreover, Eq. (25) involves three time levels. By way of contrast, the piecewise-linearized smooth method presented in the previous section provides piecewise analytical solutions in \([t_n, t_{n+1}]\) which are smooth in \([0, T]\), only involves two successive time levels for the displacement and
velocity, and yields explicit nonlinear maps for both the displacement and the velocity at the nodal points, whereas the piecewise-linearized continuous technique presented here provides only a nonlinear map for the displacement at the nodal points. It must be pointed out, however, that Eq. (10) can also be used to determine the velocity by simple differentiation of this equation.

In order to establish the interval of periodicity of Eq. (25), we consider

$$f = -\omega^2 y,$$

where $\omega \in \Re$ and the difference equation corresponding to Eq. (8) is

$$y_{n+1} - 2\phi(h^2) y_n + y_{n-1} = 0,$$

where $h = \omega k$ and $\phi(h^2) = \cos h$, and the method is zero-dissipative. The characteristic equation of the above finite difference equation can be obtained by substituting $y_n = Ar^n$ and may be expressed as

$$r^2 - 2r\phi(h^2) + 1 = 0,$$

whose roots are 1 (double) if $\omega k = 2m\pi$ where $m$ is a natural number, or complex conjugate, i.e., $r_1 = \bar{r}_1 = \exp(i\pi)$, otherwise, with $|r_1| = 1$. In addition, if $\omega k = 2m\pi$, $y_n = A + Bn$, whereas if $\omega k \neq 2m\pi$, $y_n = A\cos(n\omega k) + B\sin(n\omega k)$ which is the exact solution of Eq. (8) with $f = -\omega^2 y$, and $A$ and $B$ are constants. Therefore, the piecewise continuous method presented in this section is P-stable for all $h$ such that $|\cos h| < 1$ [51]. In addition, the method is unconditionally stable.

For $f(t, y, \dot{y}) = f(t, y)$ and $J_n = 0$, then the solution to Eq. (9) can be written as $y = A + B(t - t_n) + \frac{\beta}{2}(t - t_n)^2 + \frac{\alpha}{6}(t - t_n)^3$ which, upon applying the conditions at $t_{n-1}$, $t_n$ and $t_{n+1}$, yields Störmer’s method. It must be pointed out that an identical expression can be obtained by applying L’Hôpital rule to Eq. (25) or Eq. (26).

If $y_n \neq 0$ and a Taylor’s series expansion of the cosine in Eq. (26) is used, then this equation can be written as

$$y_{n+1} + y_{n-1} = 2y_n \left(1 + \frac{f_n}{2y_n} k^2 + O(k^4)\right),$$

which, using a Padé approximation, can be approximated by

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{f_n}{2y_n} k^2\right), \quad y_n \neq 0,$$

and coincides with the expression proposed by Li and Wu [53] and Li [54].

3. Piecewise-linearized methods for multidegree-of-freedom problems

For multidegree-of-freedom problems, i.e., $N > 1$ in Eq. (7), the piecewise-linearized methods proposed here can be generalized in different manners as indicated in the next two sections.

3.1. Matrix exponential: first-order equations I

By defining a new vector $\mathbf{z} = \dot{\mathbf{y}}$, Eq. (7) can be written as

$$\dot{\mathbf{Y}} = \mathbf{F}(t, \mathbf{Y}), \quad t > 0,$$

where $\mathbf{Y} = (y, z)^T \in \Re^{2N}$ and $\mathbf{F} \in \Re^{2N}$, subject to $y(0) = \mathbf{y}_0$ and $z(0) = \mathbf{z}_0$.

In the interval $(t_n, t_{n+1})$, Eq. (33) can be linearized with respect to $t_n$ and written as

$$\dot{\mathbf{Y}} = \mathbf{F}_n + \mathbf{T}_n (t - t_n) + \mathbf{J}_n (\mathbf{Y} - \mathbf{Y}_n), \quad t > 0,$$

where $\mathbf{F}_n = \mathbf{F}(t_n, \mathbf{Y}_n)$, $\mathbf{T}_n = \frac{d}{dt} \mathbf{F}(t_n, \mathbf{Y}_n)$ and $\mathbf{J}_n = \frac{d^2}{dt^2} \mathbf{F}(t_n, \mathbf{Y}_n)$, which represents a system of $2N$ first-order linear ordinary differential equations which can be reduced to a system of sequentially coupled equations by either applying Schur’s normal form theorem or writing $\mathbf{J}_n$ in Jordan’s canonical form. Unfortunately, in either case,
the Schur’s normal form or the Jordan canonical form have to be determined at each time step, and this is a
very expensive procedure for long-time integration.

The solution of Eq. (34) can be written as [59,60]

\[ Y(t) = \exp(\mathbf{J}_n(t-t_n))Y_n + \int_{t_n}^{t_n+1} \exp(\mathbf{J}_n(t-\tau))S d\tau, \quad t_n < t \leq t_{n+1}, \]  

(35)

where \( S \equiv \mathbf{F}_n + \mathbf{T}_n(t-t_n) \).

The integral in Eq. (35) can be performed analytically and this equation yields

\[ Y_{n+1} = \exp(\mathbf{J}_n)Y_n + Y_{pn}, \]  

(36)

where \( Y_{pn} \) denotes a particular solution.

Eq. (36) clearly indicates that the accuracy of \( Y_{n+1} \) depends on the evaluation of the matrix exponential \( \exp(\mathbf{J}_n) \). Such an evaluation can be performed efficiently by means of the \( 2^M \) algorithm which can be expressed as [61]

\[ \exp(\mathbf{J}_n) = [\exp(\mathbf{J}_n \tau)]^m, \]  

(37)

where \( \tau = \frac{k}{m}, m = 2^M \) and \( M \) is a (positive) natural number. In general, \( \tau \) must be smaller than the characteristic period of the solution, and it must be emphasized that Eq. (36) represents a system of \( 2N \) equations.

3.2. Matrix exponential: second-order equations

In this method, the piecewise-linearized method corresponding to Eq. (7) can be written as

\[ \dot{\mathbf{y}} + \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{y} = \mathbf{S}, \]  

(38)

where \( \mathbf{A} = -\frac{\mathbf{C}}{\tau^2}(t_n, \mathbf{y}_n, \dot{\mathbf{y}}_n), \mathbf{B} = \frac{\mathbf{C}}{\tau^2}(t_n, \mathbf{y}_n, \dot{\mathbf{y}}_n) \) and \( \mathbf{S} = \mathbf{f}_n + \mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{y}_n + \frac{\mathbf{C}}{\tau^2}(t_n, \mathbf{y}_n, \dot{\mathbf{y}}_n)(t-t_n) \).

If the homogeneous solution of Eq. (38) is assumed to be \( \mathbf{y}_h = \exp(\mathbf{Mt}) \) with \( \mathbf{M} = \mathbf{C} \pm \mathbf{iD} \), then

\[ \mathbf{C} = -\frac{1}{2}\mathbf{A}, \quad \mathbf{D}^2 = \frac{1}{4}\mathbf{A} + \mathbf{B}, \]  

(39)

and, therefore, \( \mathbf{D} \) is the square root of the right-hand side of the second equation of Eq. (39) and can be calculated by matrix decomposition as \( \mathbf{D} = \mathbf{Q}\sqrt{\mathbf{A}}\mathbf{Q}^{-1} \) where \( \mathbf{A} \) is a matrix consisting of the eigenvalues of \( \mathbf{D}^2 \) and \( \mathbf{Q} \) contains in column form the eigenvectors of \( \mathbf{D}^2 \). If these eigenvalues are distinct, real and different from zero, then the homogeneous solution of Eq. (38) can be written as [59,60]

\[ \mathbf{y}_h = \exp(\mathbf{A}(t-t_n))((\cos(\mathbf{D}(t-t_n)) + \mathbf{D}^{-1}\mathbf{A}\sin(\mathbf{D}(t-t_n)))\mathbf{a} + \mathbf{D}^{-1}\sin(\mathbf{D}(t-t_n))\mathbf{b}), \]  

(40)

from which it is easy to derive an expression for \( \mathbf{y}_h \), and \( \mathbf{a} \) and \( \mathbf{b} \) are constant vectors.

A particular solution of Eq. (38) can be easily derived by means of variation of parameters as

\[ \mathbf{y}_p = \mathbf{D}^{-1}\int_{t_n}^{t_n+1}\exp(\mathbf{A}(t-t_n))\sin(\mathbf{D}(t-t_n))S_n d\tau, \]  

(41)

where the integral can be performed analytically.

At \( t_{n+1} \), it is easy to show that the calculation of \( \mathbf{y}_{n+1} \) involves matrix exponentials which can be determined with the \( 2M \) algorithm described above [61,62], i.e.,

\[ \exp(i\mathbf{D}k) = [\exp(i\mathbf{D}\tau)]^m, \quad \tau = \frac{k}{m}, \quad m = 2^M, \]  

(42)

and Taylor’s series expansions

\[ \exp(i\mathbf{D}\tau) = \sum_{j=0}^{\infty} \frac{1}{j!} (i\mathbf{D}\tau)^j, \]  

(43)

which must be truncated, and the truncation should be a function of the time step, the magnitude of the absolute values of the eigenvalues of \( \mathbf{D} \) and the magnitude of the desired truncation error tolerance.
A particular case of the method presented in this section has been previously developed by the author for $A = 0$, i.e., $f(t, y, \dot{y}) = f(t, y)$, and for the case that the eigenvalues of $B$ are positive, real and distinct [63].

3.3. Matrix exponential: first-order equations II

The methods presented in the two previous sections require the evaluation of matrix exponentials and are based on the piecewise-linearization of Eq. (7) written as a system of either first- or second-order differential equations that contain the first-order derivative. These methods are subject to the errors incurred by the piecewise local linearization and those associated with the evaluation of the matrix exponential.

In this section, we describe a method for systems of first-order ordinary differential equations which does not make use of piecewise linearization and, therefore, is not subject to the errors associated with local linearization.

Eq. (7) may be written as

$$\dot{Y} = LY + N,$$

where $L$ is here a constant matrix, i.e., a linear operator, and $N = F - LY$, whose solution is

$$\dot{Y}(t) = \exp(Lt)Y_0 + \int_0^t \exp(L(t - \tau))N\,d\tau. \quad (45)$$

The integral in Eq. (45) may be approximated by a quadrature rule and this equation can, therefore, be approximated by

$$Y_{n+1} = \exp(Jnk)Y_n + k \left( a N_{n+1} + \sum_{j=0}^{M-1} \beta_j N_{n-j} \right), \quad (46)$$

where $M + 1$ is the number of time levels involved in the discretization, and $\alpha$ and $\beta_j$ are to be determined by expanding $Y_{n+1}$ in Taylor series around $t_n$, substituting the expansion into Eq. (46) and equating terms in like powers of $k$ which yield, after lengthy algebra,

$$\alpha S_{n+1} + \sum_{j=0}^{M-1} \beta_j S_{n-j} = \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\partial S}{\partial Y} \right)_n k^i \left( \alpha + (-1)^i \sum_{j=0}^{M-1} \beta_j j^i \right), \quad (47)$$

$$\alpha + \sum_{j=0}^{M-1} \beta_j = R_1, \quad (48)$$

and

$$\frac{1}{k!} \left( \alpha + (-1)^k \sum_{i=1}^{M-1} i^k \beta_i \right) = R_{k+1}, \quad k = 1, 2, \ldots, M, \quad (49)$$

where

$$R_j = \frac{1}{x^j} \left( e^x - \sum_{k=0}^{j-1} \frac{1}{k!} x^k \right). \quad (50)$$

As indicated above, the key point of this technique is the evaluation of

$$Q_j(Lk) = \frac{\exp(Lk) - E_j(Lk)}{(Lk)^j}, \quad E_j(Lk) = \sum_{m=0}^{j-1} \frac{(Lk)^m}{j!} \quad (51)$$

for $j = 0, 1, \ldots$

These terms could be evaluated by means of Taylor’s series expansions, but this may result in a loss of accuracy. To avoid this problem, the matrix exponential can be calculated by means of the scaling-and-squaring algorithm which is based on the identity $Q_0(2x) = 2Q_0(x)$ and the use of the $2M$ technique describe previously.
4. Presentation of results

The accuracy of the piecewise-linearized smooth and piecewise-linearized continuous methods presented in the previous section has been assessed by comparing their numerical solutions with exact ones when available for the examples below. It must be emphasized that the piecewise-linearized methods presented in this paper were implemented as indicated in Section 2 without employing any Taylor’s series approximation when, for example $J_n \approx 0$ in Eq. (26), in order to illustrate their robustness. If such a series approximations had been made, an improvement over the accuracy reported here might have been achieved.

**Example 1.** This example corresponds to

$$\ddot{y} = -\omega^2 y + (\omega^2 - 1) \sin t,$$

with $y(0) = 1$ and $\dot{y}(0) = 1 + \omega$, which has the following exact solution:

$$y(t) = \cos(\omega t) + \sin(\omega t) + \sin t,$$

which clearly shows the presence of two frequencies [46].

Some sample results are illustrated in Table 1 which shows the largest, i.e., $L_\infty$-norm, errors for $[0, 100]$, i.e.,

$$\max_{t\in[0,100]} |y_n - y_e(nk)|,$$

for $\omega = 10$ using time steps equal to $k = (\frac{1}{2})^m$, where the subscript $e$ denotes exact solution. In this table as well as in the other ones presented in this paper, PS and PC refer to the piecewise-linearized (P) smooth (S) and continuous (C), respectively, methods described in Section 2, while PSN refers to the PS method that results from not performing the linearization of $f$ with respect to $t$, i.e., it is the PS method that results from setting $T_n = 0$ in Eq. (9).

Table 1 shows that the accuracy of PS is higher than that of PSN which is, in turn, more accurate than PC. The errors of the latter are very large for large time steps and this method exhibits a second-order convergence rate whereas that of PS and PSN is only of first order. Table 1 also shows that, for this example, the linearization with respect to $t$ affects considerably the accuracy of piecewise-linearized smooth methods.

The results of PS presented in Table 1 are more accurate than those of RKN2-10 and less accurate than those of RKN2-4, RKN4-4 and the trigonometrically-fitted Runge–Kutta–Nyström (RKN) method reported by Paternoster [46] who obtained the solution in $[0, 20\pi]$ and showed that only $\omega = 10$ was fitted in his trigonometrically-fitted technique.

**Example 2.** This example corresponds to

$$\ddot{y} + y + y^3 = (\cos t + \epsilon \sin t)^3 - 99\epsilon \sin(10t),$$

with $y(0) = 1$ and $\dot{y}(0) = 10\epsilon$, which has the following exact solution:

$$y(t) = \cos(t) + \epsilon \sin(10t),$$

which clearly shows the presence of two frequencies [21].

---

**Table 1**

$L_\infty$-norm errors for Example 1 with $k = (\frac{1}{2})^m$

<table>
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<tr>
<th>$m$</th>
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<th>PSN</th>
<th>PC</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.8629E+00</td>
<td>4.4496E+07</td>
</tr>
<tr>
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<td>2.8385E+00</td>
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<td>2.8385E+00</td>
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<td>1.3671E+00</td>
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<td>0.3568E+00</td>
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<td>8</td>
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<td>0.0894E+00</td>
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<td>0.0224E+00</td>
</tr>
<tr>
<td>10</td>
<td>3.9219E-07</td>
<td>9.7622E-04</td>
<td>0.0056E+00</td>
</tr>
</tbody>
</table>
Table 2 illustrates the results obtained with the piecewise linear methods presented in this paper for \([0, 100]\) and \(\epsilon = 0.001\). This table reveals that PS and PC are second-order accurate, whereas PSN is only first-order accurate. Table 2 also shows the results obtained with the Störmer–Cowell–Verlet method, here referred to as ST, i.e.,

\[ y_{n+1} - 2y_n + y_{n-1} = k^2 f_n, \]

which is a second-order accurate, non-self-starting, explicit finite difference equation for Eq. (8) without \(y\). For Example 2, ST is second-order accurate and was started as PC.

Table 2 also indicates that ST is more accurate than the piecewise-linearized methods presented in this paper. Although not shown here, PCN which is identical to PC but without the term \(T_n\), yielded analogous results to those of PC.

The results presented in Table 2 are in good accord with those reported by Franco [21] who determined the maximum errors of its embedded trigonometrically-fitted Runge–Kutta method implemented in a variable step manner as a function of the number of function evaluations.

Calculations have also been performed with the following technique:

\[ y_{n+1} - 2y_n + y_{n-1} = k^2 (xf_{n-1} + (1 - 2x)f_n + xf_{n+1}), \]

where \(x = 0\) corresponds to the explicit Störmer method, \(x = \frac{1}{12}\) yields the implicit fourth-order accurate Numerov or compact method, and \(x = \frac{1}{2}\) and \(x = \frac{1}{4}\). The values of \(x = \frac{1}{12}, \frac{1}{4}\) and \(\frac{1}{2}\) yield finite difference methods which are here referred to as N, ST1/4 and ST1/2, respectively; these methods are implicit and their finite difference equations were solved iteratively until the following convergence criterion was satisfied:

\[ |y_{n+1}^{k+1} - y_{n+1}^k| \leq 10^{-6}, \]

where the superscript \(k\) denotes the \(k\)th iteration within the time step. In addition, for \(x \neq 0\), and in order to avoid iterations, Eq. (57) was linearized with respect to the \(n\)-th time level and, for \(f(t, y, \dot{y}) = f(t, y)\), can be written as

\[ y_{n+1} - 2y_n + y_{n-1} = k^2 (xf_{n-1} + (1 - 2x)f_n + x(f_n + T_n k + J_n (y_{n+1} - y_n))), \]

where \(f_n = f(t_n, y_n)\) and \(J_n = \frac{\partial f}{\partial y}(t_n, y_n)\).

The linearized methods corresponding to Eq. (59) and the values of \(x = \frac{1}{12}, \frac{1}{4}\) and \(\frac{1}{2}\) are here referred to as NL, STL1/4 and STL1/2, respectively.

Table 3 shows that N is more accurate than ST1/4 which, in turn, is more accurate than ST1/2. The accuracy of STL1/4 is lower than that of ST1/4. In any case, N, ST1/4, ST1/2 and STL1/4 exhibit second-order accuracy for Example 2.

Example 3. This example corresponds to

\[ \ddot{y} + 4r^2 \dot{y} = (4r^2 - \omega^2) \sin(\omega t) - 2 \sin(r^2), \]
with \( y(0) = 1 \) and \( \dot{y}(0) = \omega \), which has the following exact solution:

\[
y(t) = \cos(t^2) + \sin(\omega t),
\]

which clearly exhibits the presence of two frequencies [21].

Table 4 illustrates the results obtained with the piecewise-linearized methods presented in this paper for \([0, 50]\) and \( \omega = 10 \). This table clearly shows that very tiny time steps are required by the piecewise-linearized methods presented in this paper to obtain accurate results. The reason for this behavior was found to be associated with the fact that \( J \) was nil at \( t = 0 \) and the term \( \frac{\partial y}{\partial t} \). Therefore, we developed a new piecewise smooth method where \( f \) was expanded about \( (t_{n+1/2}, y_n, \dot{y}_n) \) which is here referred to as PSH. As Table 4 indicates, the behavior of this method is very similar to that of PS, and PSN and PCN were found to be more accurate than PS and PSH, for small time steps. Table 4 also shows that PSHN which is identical to PSH except that the term \( \frac{\partial y}{\partial t} \) is set to zero, is the most accurate technique for small time steps. For large time steps, the effects of \( J = 0 \) at \( t = 0 \) are visible for all the piecewise linearization methods presented in this paper.

Table 4 shows that the piecewise-linearized methods presented in this paper provide less accurate results than those reported by Franco [21] who used an embedded trigonometrically-fitted Runge–Kutta method implemented in a variable step manner. However, it must be pointed out that an improvement of the accuracy of PS can be easily achieved by employing a variable time step whose value is determined as a function of \( |y_{n+1} - y_n| \) and \( J_n \), or an error tolerance.

**Example 4.** This example corresponds to

\[
\ddot{y} + \omega^2 y = (\omega^2 - 4t^2) \cos(t^2) - 2 \sin(t^2),
\]

with \( y(0) = 1 \) and \( \dot{y}(0) = \omega \), which has the following exact solution:

\[
y(t) = \cos(t^2) + \sin(\omega t),
\]

which clearly exhibits the presence of two frequencies [23].
Table 5 illustrates the results obtained with the piecewise-linearized methods presented in this paper for \([0, 100]\) and \(x = 100\). This table shows that the accuracy of PCN is comparable to that of PS but much higher than that of PSN, for small time steps; for large time steps, these three methods are very inaccurate.

The results presented in Table 5 are more accurate than those obtained with a fourth-order algebraic predictor–corrector scheme, but less accurate than those of a classical fourth-order algebraic Runge–Kutta method, an embedded Runge–Kutta Dormand–Price scheme, and a fourth-order algebraic trigonometrically-fitted predictor–corrector algorithm [23]. This is not surprising, for a fixed step size was employed in the piecewise-linearized techniques reported in this paper.

**Example 5.** This example corresponds to

\[
\dot{y} = -100y + 99 \sin(t),
\]

with \(y(0) = 1\) and \(\dot{y}(0) = 11\), which has the following exact solution:

\[
y(t) = \cos(10t) + \sin(10t) + \sin(t),
\]

which clearly exhibits the presence of two frequencies [23].

Table 6 illustrates the results obtained with the piecewise-linearized methods presented in this paper for \([0, 100]\) and \(\omega = 100\). This table shows that PS is more accurate than PCN which, in turn, is more accurate than PSN. PS and PCN are second-order accurate, whereas PSN is only first-order accurate.

The results of PS presented in Table 6 are more accurate than those obtained with a fourth-order algebraic predictor–corrector scheme, a classical fourth-order algebraic Runge–Kutta method, an embedded Runge–Kutta Dormand–Price scheme, and a fourth-order algebraic trigonometrically-fitted predictor–corrector algorithm [23], whereas those of PSN and PCN are more accurate than those obtained with a fourth-order algebraic predictor–corrector scheme, a classical fourth-order algebraic Runge–Kutta method and an embedded Runge–Kutta Dormand–Price scheme [23]. PS is also more accurate than the exponentially-fitted symplectic methods [a] and [b] of Simos and Vigo-Aguiar [32].
Example 6. This example corresponds to
\[ \dot{y} = -\omega^2 y, \]
with \( y(0) = 1 \) and \( \dot{y}(0) = 0 \), which has the following exact solution:
\[ y(t) = \cos(t). \]  

Table 7 illustrates the results obtained with the piecewise-linearized methods presented in this paper for \([0, 100]\) and \( \omega = 1 \). This table shows that PS is more accurate than PC and is not very much affected by the step size. This is a consequence of the numerical evaluation of the trigonometric functions, for PS and PSN provide the exact solution to Eq. (66) in exact arithmetic.

The relatively large errors of PCN which coincides with PC for the larger step sizes are a consequence of the use of a fixed step size and the determination of the initial conditions for starting this technique. As shown in this table, the errors of PCN decrease very rapidly as the step size is decreased, until the numerical evaluation of the trigonometric functions introduces some saturation.

Example 7. This example was proposed by Stiefel and Bettis [55] and corresponds to
\[ \dot{y} + y = 0.001 \exp(it), \]
with \( y(0) = 1 \) and \( \dot{y}(0) = 0.9995i \), which has the following exact solution:
\[ y(t) = \exp(it) - 0.0005t \exp(it), \] and \( y = u + iv \), where \( u \) and \( v \) are real variables.

Tables 8 and 9 illustrate the results obtained with the piecewise-linearized methods presented in this paper for \([0, 100]\). This table shows that the accuracy of PCN is comparable to that of PC but higher than that of PS. For this example, PSN was found to be the least accurate piecewise-linearized technique. In addition, Table 7

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</table>
and 9 indicate that PS, PC and PCN are second-order accurate techniques, whereas PSN is only first-order accurate.

The results presented in Tables 8 and 9 indicate that the piecewise-linearized methods PS, PC and PCN presented in this paper are slightly less accurate than those obtained with a four-step trigonometrically-fitted P-stable Obrechkoff method [44] for the largest time steps considered in this table; this Obrechkoff technique has an interval of periodicity equal to about (0, 16) and the maximum-norm errors predicted by this technique increase as $t$ increases. For this example, piecewise-linearized methods are also less accurate than exponentially-fitted techniques that integrate exactly any linear combination of the functions $[1, t, \ldots, t^8, \exp(\pm\omega t)]$ [31] and exponentially-fitted explicit Runge–Kutta algorithms [18]. On the other hand, PS, PC and PCN are more accurate than Simos’ dissipative trigonometrically-fitted technique [33], trigonometrically-fitted predictor–corrector techniques [24], and exponentially-fitted symplectic integrators [32].

**Example 8.** This example corresponds to

$$\ddot{y} + y + y^3 = B \cos(\omega t),$$  \hfill (70)

with $y(0) = 0.20042672806900$ and $\dot{y}(0) = 0$, which has the following very accurate approximate solution for $B = 0.002$:

$$y(t) = 0.200179477536 \cos(\omega t) + 0.246946143 \times 10^{-3} \cos(3\omega t) + 0.304016 \times 10^{-6} \cos(5\omega t) + 0.374 \times 10^{-9} \cos(7\omega t).$$  \hfill (71)

Table 10 illustrates the results obtained with the piecewise-linearized methods presented in this paper for $[0, 100]$ and $\omega = 1.01$, and shows that PS, PC and ST are second-order accurate, whereas PSN is only first-order accurate. Table 10 also shows that PS is more accurate than PSN, PC and ST. ST is more accurate than PC, and the least accurate piecewise-linearized method is PSN.

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**Table 9**

$L_{1\infty}$-norm errors in $v$ for Example 7 with $k = (\frac{1}{2})^m$ and $t \in [0, 100]$

<table>
<thead>
<tr>
<th>$m$</th>
<th>PS</th>
<th>PSN</th>
<th>PC</th>
<th>PCN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0020</td>
<td>0.0122</td>
<td>0.0429</td>
<td>0.0429</td>
</tr>
<tr>
<td>2</td>
<td>5.0648E-04</td>
<td>0.0062</td>
<td>0.0105</td>
<td>0.0105</td>
</tr>
<tr>
<td>3</td>
<td>1.2655E-04</td>
<td>0.0031</td>
<td>0.0026</td>
<td>0.0026</td>
</tr>
<tr>
<td>4</td>
<td>3.1670E-05</td>
<td>0.0015</td>
<td>6.5066E-04</td>
<td>6.5066E-04</td>
</tr>
<tr>
<td>5</td>
<td>7.9403E-06</td>
<td>7.7318E-04</td>
<td>1.6260E-04</td>
<td>1.6260E-04</td>
</tr>
<tr>
<td>6</td>
<td>1.9837E-06</td>
<td>3.8660E-04</td>
<td>4.0630E-05</td>
<td>4.0630E-05</td>
</tr>
<tr>
<td>7</td>
<td>4.9592E-07</td>
<td>1.9329E-04</td>
<td>1.0140E-05</td>
<td>1.0140E-05</td>
</tr>
<tr>
<td>8</td>
<td>1.2610E-07</td>
<td>9.6651E-05</td>
<td>2.5174E-06</td>
<td>2.5174E-06</td>
</tr>
<tr>
<td>9</td>
<td>2.9902E-08</td>
<td>4.8322E-05</td>
<td>6.1187E-07</td>
<td>6.1187E-07</td>
</tr>
<tr>
<td>10</td>
<td>8.2367E-09</td>
<td>2.4163E-05</td>
<td>1.3543E-05</td>
<td>1.3542E-07</td>
</tr>
</tbody>
</table>

**Table 10**

$L_{1\infty}$-norm errors for Example 8 with $k = (\frac{1}{2})^m$

<table>
<thead>
<tr>
<th>$m$</th>
<th>PS</th>
<th>PSN</th>
<th>PC</th>
<th>ST</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0327</td>
<td>0.2080E+00</td>
<td>0.0799</td>
<td>0.1542</td>
</tr>
<tr>
<td>2</td>
<td>0.0072</td>
<td>0.0272E+00</td>
<td>0.0401</td>
<td>0.0403</td>
</tr>
<tr>
<td>3</td>
<td>0.0016</td>
<td>0.0067E+00</td>
<td>0.0202</td>
<td>0.0102</td>
</tr>
<tr>
<td>4</td>
<td>3.7664E-04</td>
<td>0.0017E+00</td>
<td>0.0101</td>
<td>0.0025</td>
</tr>
<tr>
<td>5</td>
<td>9.0999E-05</td>
<td>0.0243E+00</td>
<td>0.0050</td>
<td>6.3627E-04</td>
</tr>
<tr>
<td>6</td>
<td>2.2258E-05</td>
<td>0.0122E+00</td>
<td>0.0025</td>
<td>1.5907E-04</td>
</tr>
<tr>
<td>7</td>
<td>5.4106E-06</td>
<td>0.0061E+00</td>
<td>0.0013</td>
<td>3.9765E-05</td>
</tr>
<tr>
<td>8</td>
<td>1.2416E-06</td>
<td>0.0030E+00</td>
<td>6.2877E-04</td>
<td>9.9376E-06</td>
</tr>
<tr>
<td>9</td>
<td>2.0338E-07</td>
<td>0.0015E+00</td>
<td>3.1435E-04</td>
<td>2.4807E-06</td>
</tr>
<tr>
<td>10</td>
<td>5.5227E-08</td>
<td>7.6081E-04</td>
<td>1.5717E-04</td>
<td>6.1653E-07</td>
</tr>
</tbody>
</table>
Table 10 indicates that PS is less (more) accurate than the exponentially-fitted symplectic methods [b] ([a]) of Simos and Vigo-Aguiar [32]; it is also less accurate than exponentially-fitted explicit Runge–Kutta techniques [18] and four-step trigonometrically-fitted P-stable Obrechkoff algorithms [44].

Table 11 shows that N is more accurate than ST1/4 which, in turn, is more accurate than ST1/2. The accuracy of STL1/4 is lower than that of ST1/4. In any case, N, ST1/4, ST1/2 and STL1/4 exhibit second-order accuracy for Example 8.

A comparison between Tables 10 and 11 indicates that N is more accurate than the piecewise-linearized methods presented in this paper, for Example 8. This is not surprising in view of the fact that the Numerov method is fourth-order accurate, whereas one can show by means of Taylor’s series expansions that the accuracy of piecewise-linearized methods is at most second-order. On the other hand, Numerov’s technique and, in fact, any method derived from Eq. (57) with $z \neq 0$, results in a nonlinear algebraic equation that has to be solved iteratively, whereas the piecewise-linearized methods presented in this paper provide explicit finite difference formulae for both the displacement and velocity in PS algorithms or for the displacement in PC schemes. In addition, the Numerov method provides only nodal (discrete) values and interpolation is required to determine the displacement, whereas piecewise-linearized methods provide piecewise analytical solutions for the displacement which are either smooth everywhere for PS or continuous everywhere for PC. In addition, it has been shown in this paper that PC is P-stable in open intervals that do not include $h = 2\pi m$ where $m$ is a natural number, whereas it is an easy exercise that the Numerov method is only P-stable for $h^2 \leq 6$. By way of contrast, ST is P-stable for $h^2 \leq 4$; ST1/4 and ST1/2 are always P-stable, but they are less accurate than the Numerov technique.

As indicated in Section 2, piecewise-linearized smooth methods can employ variable time steps in a straightforward manner and are self-starting. Piecewise-linearized continuous techniques may be generalized to account for variable time steps by means of a generalization of the method presented in Section 2, but are not self-starting. Although the two kinds of piecewise-linearized algorithms presented in this paper have been applied to two degrees-of-freedom problems, i.e., second-order ordinary differential equations, they may generalized to systems of first-order ordinary differential equations by introducing a new variable $z \equiv \dot{y}$.

Example 9. This example corresponds to

$$\ddot{x} + cx + x^3 = -\cos(t) - c \sin(t) + \cos^3(t),$$

with $x(0) = 1$ and $\dot{x}(0) = 0$, which has the following exact solution:

$$x(t) = \cos(t).$$

This problem has been considered previously with $c = 0.05$ [56] and it has been observed that the solution of Eq. (72) is “unstable” regardless of the adopted integration method and of the time step, where the term “unstable solution” means that the numerical solution of Eq. (72) differs from the exact solution after some time. This behavior has also been observed when Newmark’s time integration method is employed to solve Eq. (72) [57], and may be attributed to the presence of spurious fixed points or attractors of the finite difference equations [58]. In fact, it is known that consistent energy dissipative schemes exhibit no spurious solutions,

<table>
<thead>
<tr>
<th>$m$</th>
<th>N</th>
<th>ST1/2</th>
<th>ST1/4</th>
<th>STL1/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0026E+00</td>
<td>0.4375E+00</td>
<td>0.3113E+00</td>
<td>0.4362E+00</td>
</tr>
<tr>
<td>2</td>
<td>2.0302E-04</td>
<td>0.2033E+00</td>
<td>0.0815E+00</td>
<td>0.2166E+00</td>
</tr>
<tr>
<td>3</td>
<td>2.0157E-05</td>
<td>0.0511E+00</td>
<td>0.0204E+00</td>
<td>0.0530E+00</td>
</tr>
<tr>
<td>4</td>
<td>2.3207E-06</td>
<td>0.0127E+00</td>
<td>0.0051E+00</td>
<td>0.0132E+00</td>
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<tr>
<td>5</td>
<td>2.8846E-07</td>
<td>0.0032E+00</td>
<td>0.0013E+00</td>
<td>0.0033E+00</td>
</tr>
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<td>6</td>
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<td>3.1813E-04</td>
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<td>6.5741E-09</td>
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<td>5.1487E-05</td>
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<td>9</td>
<td>8.0727E-09</td>
<td>1.2433E-05</td>
<td>4.9758E-06</td>
<td>1.2877E-05</td>
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<tr>
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<td>3.1099E-06</td>
<td>1.2467E-06</td>
<td>3.2215E-06</td>
</tr>
</tbody>
</table>
whereas methods with unconditional phase-space conservation and the trapezoidal rule may have spurious steady states [58].

Fig. 1 shows the results obtained with PS for different time steps, and indicates that, for $k = 0.5$, there are large discrepancies between the analytical and numerical solutions. These solutions are in good accord with each other for up to $t = 40, 40$ and $10$ for $k = 0.01, 0.005$ and $0.001$, respectively; at larger times, the numerical solution differs from the analytical one and these differences are functions of the time step.

Fig. 2 shows the results obtained with PSN and indicates that the agreement between the analytical and numerical solutions improves as the time step is decreased; but, even for $k = 0.001$, the numerical solution “abandons” the exact one at about $t = 40$.

If the linearization of Eq. (72) is performed with respect to $t_{n+1}$, but the term $\frac{\partial y}{\partial t}$ is set to zero (cf. Eq. (9)), one obtains the results illustrated in Fig. 3 which indicate that the agreement between the analytical and numerical solutions improves as the time step is decreased; but, even for $k = 0.001$, the numerical solution abandons the exact one at about $t = 40$. On the other hand, if the linearization of Eq. (72) is performed with respect to $t_{n+\frac{1}{2}}$, one obtains similar trends to those of Figs. 2 and 3, but good agreement between the exact and numerical solutions can be observed up to approximately $t = 75$ for $k = 0.001$ as indicated in Fig. 4.

For $c = 0$, the results presented in Figs. 5–8 indicate that the agreement between the exact solutions and the numerical ones of PS increases as the time step is decreased. The time at which the numerical solution abandons the exact one for $k = 0.001$ is about 80, 40, 40 and 80 for PS, PSN, PSN with linearization about $t_{n+1}$, and PSN with linearization about $t_{n+\frac{1}{2}}$, respectively. For large time steps, the results presented in Figs. 5–8 show that these methods produce erratic oscillations and the numerical solution may be very close to the exact one for some time intervals as illustrated, for example, in Figs. 5 (top left), 6 (top right) and 8 (top left).

The numerical results shown in Figs. 1–8 exhibit the same trends as those of the Newmark’s method [57], and indicate that the accuracy order, the stability range, the spurious fixed points/attractors and the dissipa-
Fig. 2. $x(t)$ for Example 9 with $c = 0.05$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PSN. Solid line: numerical solution; dashed line: exact solution.

Fig. 3. $x(t)$ for Example 9 with $c = 0.05$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PSN and linearization with respect to $t_{n+1}$. Solid line: numerical solution; dashed line: exact solution.
Fig. 4. $x(t)$ for Example 9 with $c = 0.05$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PSN and linearization with respect to $t_{n+1}$. Solid line: numerical solution; dashed line: exact solution.

Fig. 5. $x(t)$ for Example 9 with $c = 0$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PS. Solid line: numerical solution; dashed line: exact solution.
Fig. 6. $x(t)$ for Example 9 with $c = 0$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PSN. Solid line: numerical solution; dashed line: exact solution.

Fig. 7. $x(t)$ for Example 9 with $c = 0$ for $k = 0.5$ (top left), 0.01 (top right), 0.005 (bottom left) and 0.001 (bottom right) obtained with PSN and linearization with respect to $t_{n+1}$. Solid line: numerical solution; dashed line: exact solution.
tion/conservation properties of numerical methods play an important role in determining their long-time accuracy.

5. Conclusions

Two piecewise-linearized methods based on the Taylor series expansion of the right-hand side of second-order ordinary differential equations written in canonical form have been developed for nonlinear oscillators with periodic solutions when the differential equation contains the first-order derivative. The methods are based on the Taylor series expansion of the nonlinearities with respect to time and the displacement and velocity, and, for single degree-of-freedom problems, provide piecewise analytical solutions in either two- or three-point intervals which are continuous everywhere. One of the methods provides explicit two-point finite difference equations for both the displacement and the velocity at nodal points, and smooth solutions everywhere, and is self-starting. The other provides explicit three-point finite difference equations for the displacement only and continuous solutions everywhere, but it is not self-starting. Both methods have an infinite interval of periodicity and are P-stable.

It has been shown that the piecewise-linearized continuous method for single degree-of-freedom problems reduces to the well-known Störmer–Cowell–Verlet technique if quartic and higher-order terms of the Taylor series expansion of the cosine function are neglected.

It has also been demonstrated that a Taylor series expansion of the cosine function together with the use of a Padé approximation result in a P-stable method that has been postulated previously, when the Jacobian is not nil. Although, the resulting method is also P-stable, it does not provide the exact solution of linear, homogeneous, second-order differential equations with no damping and a constant natural frequency.

Two three-point iterative and two three-point time-linearized methods have also been developed. These techniques are generalizations of the Numerov scheme and are not self-starting, and their periodicity intervals have been analyzed.
The accuracy of the piecewise-linearized smooth and continuous methods has been assessed by comparing their numerical solutions with analytical ones as a function of the time step, and it has been shown that, in general, the piecewise-linearized smooth technique is more accurate than the piecewise-linearized continuous one. The accuracy of the former depends on the nonlinearities and time-dependent forcing terms, and, in some cases, may be improved by either neglecting the terms that depend on the partial derivative of the forcing term with respect to the time variable or performing the linearization with respect to the midpoint of the interval of integration.

Although the piecewise-linearized methods provide piecewise-analytical solutions which are either exponential or trigonometric, they do not require an estimate of the dominant frequency; by way of contrast, exponentially-fitted methods do require such an estimate and are based on the discretization of the second-order differential equation. However, except in some examples, exponentially-fitted methods provide more accurate solutions than piecewise-linearized algorithms, unless the latter employ variable time steps that are adjusted as the numerical solution evolves. On the other hand, piecewise-linearized methods have been formulated for second-order differential equations that contain first-order derivatives, whereas exponentially-fitted techniques have been developed for second-order differential equations which do not contain first-order derivatives.

It has been shown that the accuracy of piecewise-linearized techniques depends very much on the nonlinearities, time step and forcing terms, and on the existence of spurious fixed points/attractors in the resulting finite difference equations. In one of the examples considered in this paper, it has been shown that the numerical solution follows closely the analytical one up to certain time, beyond which the numerical solution abandons the exact one and may exhibit very erratic behavior characterized by multiple frequencies, chaos, etc.

The piecewise linearization methods presented here have also been generalized for multiple degree-of-freedom problems which contain the velocity and piecewise analytical solutions which are continuous everywhere have been obtained. This generalization also results in explicit three-point finite difference equations.

Acknowledgement

The research reported in this paper was partially supported by Project FIS2005-03191 from the Ministerio de Educación y Ciencia of Spain and fondos FEDER.

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Z. Wang, Y. Wang, A new kind of high-efficient and high-accurate P-stable Obrechkoff three-step method for periodic initial-value
Z. Wang, Q. Chen, A trigonometrically-fitted one-step method with multi-derivative for the numerical solution of the one-
K. Lorenz, T. Jahnke, Ch. Lubich, Adiabatic integrators for highly oscillatory second-order linear differential equations with time-