Traction boundary elements for cracks in anisotropic solids

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Abstract

A general mixed boundary element approach based on displacement and traction integral equations for anisotropic media is presented. Integration of the singular and hypersingular kernels along general quadratic line elements is carried out by analytical transformation of the integrals into regular ones, which are numerically evaluated, plus simple singular integrals with known analytical solution. This is achieved by the simple election of an integration variable, which is consistent with that of the anisotropic fundamental solution. The generality of the method allows for the use of curved elements and discontinuous quarter-point elements to represent Fracture Mechanics problems. Stress Intensity Factors are accurately computed from the crack opening displacement at the nodes of the quarter-point element. Several examples, including curved crack geometries and different material properties are presented.

Keywords: Boundary element method; Hypersingular; Anisotropic media; Fracture mechanics; Stress intensity factor; Quarter-point element

1. Introduction

The extensive use of composite materials and the subsequent need of their structural integrity analysis has led to the development of different numerical tools for crack problems in anisotropic materials. Boundary Element (BE) formulations for those materials were presented in the early years of the method by Rizzo and Shippy [1], Tomlin and Batterfield [2], Cruse and Swedlow [3] and Snyder and Cruse [4]. Cruse and his coworkers [4,5] presented the anisotropic integral equation formulation and combined it with a special fundamental solution (Green’s function) which incorporates the existence of a traction-free crack.

More recently, new BE formulations appropriate for crack problems have been extended to anisotropic materials. First, formulations based on the classical displacement integral representation and the subdomain technique, as those presented by Tan and Gao [6] in 1992, and by Sollero and Aliabadi [7] in 1993 for two-dimensional (2D) fully anisotropic materials, and the one presented by Sáez et al. [8] in 1997 for three-dimensional transversely isotropic solids. Later, hypersingular formulations, based on the tractions integral representation, and mixed or dual formulations, have been presented.

The Dual BE formulation for 2D anisotropic solids was first reported by Sollero [9] in 1994 and Sollero and Aliabadi [10] in 1995. They used a combination of the displacement and traction integral representations to solve crack problems with discretization only of the crack surfaces and the external boundary. Hypersingular integrals were computed using a regularization approach previously reported by Portela et al. [11], valid for straight line elements. Sollero and Aliabadi [10] computed Stress Intensity Factors (SIF) by means of the $J$-integral technique. A similar mixed formulation was presented by Pan and Amadei [12] in 1996, extending its use to curved elements. They computed the hypersingular integrals along those elements by using a Gauss quadrature for Finite Parts integrals developed by Tsamasphyros and Dimou [13]. Pan [14] extended this mixed formulation by computing the SIF from the crack surfaces displacements using an extrapolation method.

In the present paper, a new BE approach based on the mixed formulation, for the analysis of 2D crack problems in anisotropic solids is presented. The tractions integral representation is written for the crack surfaces and the displacements integral representation for the external...
boundaries. Strongly singular and hypersingular integrals appearing in the formulation are transformed into regular ones by a simple analytical procedure valid for elements of arbitrary geometry. The basic variables are the crack opening displacements (COD) on the crack surfaces and the displacements and tractions on the external boundaries. Quadratic quarter-point element are used to represent the COD near the crack tip as done by Sáez et al. [15] for the isotropic case. The SIF are computed in a direct way from the COD at a point extremely close to the crack tip, yielding very accurate results.

The present formulation represents an improvement with respect to that presented by Sollero and Aliabadi [10] since it is valid not only for straight line crack elements with constant value of the Jacobian, but for general elements, including curved elements and quarter-point crack tip elements which are useful for the direct and simple computation of the SIF. It is also different from the formulation proposed by Pan and Amadei [12] and Pan [14], since in the present one, there are no hypersingular integrals to be numerically computed. These integrals are analytically transformed into regular ones by the simple selection of an integration variable, which is consistent with that of the anisotropic fundamental solution. Besides, the SIF computation approach is also different, simple and very accurate. The integration approach can be easily extended to other static and dynamic anisotropic problems such as half-plane domains and piezoelectric materials, as will be shown in a forthcoming paper.

2. Basic equations and BE formulation

The mixed formulation for the BE solution of crack problems is based on both displacement and traction integral representations. In the case of zero body forces, the 2D displacement integral representation for a point with coordinates \( \xi_k \), \( \xi_2 \) can be written as

\[
 c_{ij}(\xi) u_j(\xi) + \int_r p^i_j(x, \xi) u_j(x) d\Gamma(x) = \int_r u^i_j(x, \xi) p_j(x) d\Gamma(x)
\]

(1)

where \( i, j = 1, 2 \); \( u^i_j \) and \( p^i_j \) are the fundamental solution displacements and tractions, respectively; \( c_{ij}(\xi) = \delta_{ij} \) where \( \xi \) is an internal point and \( c_{ij}(\xi) = \frac{1}{2} \delta_{ij} \) for a smooth boundary point. In what follows, the source point \( \xi \) and the observation point \( x \) are defined on the complex plane as

\[
z^*_k = \xi_1 + \mu_k \xi_2; \quad z^*_k = x_1 + \mu_k x_2, \quad k = 1, 2
\]

\( \mu_k \) being the roots of the characteristic equation of the anisotropic material

\[
a_{11} \mu^4 - 2a_{16} \mu^3 + (2a_{12} + a_{66}) \mu^2 - 2a_{26} \mu + a_{22} = 0
\]

(3)

\( a_{ij} (i, j = 1, 2, 6) \) being the compliance coefficients. The roots of Eq. (3) are either complex or purely imaginary and always occur in conjugate pairs [16].

The fundamental solution displacements and tractions are given by [7]

\[
u^i_j(z_k^*, z_l^*) = 2 \text{Re} \left\{ \sum_{k=1}^2 \text{Re} \left\{ P_{jk} A_k \ln(z_k^* - z_l^*) \right\} \right\}
\]

(4)

\[
p^i_j(z_k^*, z_l^*) = 2 \text{Re} \left\{ \sum_{k=1}^2 Q_{jk}(\mu_k n_1 - n_2) \frac{A_k}{z_k^* - z_l^*} \right\}
\]

(5)

where \( n \) is the outward unit normal at the observation point \( x \) and

\[
\left\{ \begin{array}{l} P_{ik} \\ P_{jk} \end{array} \right\} = \left\{ \begin{array}{l} a_{11} \mu_k^2 + a_{12} - a_{16} \mu_k \\ a_{12} \mu_k + a_{22} / \mu_k - a_{26} \end{array} \right\}
\]

(6)

\[
Q_{jk} = \mu_k \delta_{1j} - \delta_{2j}
\]

(7)

\[
\left( \begin{array}{ccc} 1 & -1 & 1 \\ \mu_1 & -\mu_1 & \mu_2 - \mu_2 \\ P_{11} & -P_{11} & P_{12} \\ P_{21} & -P_{21} & P_{22} \end{array} \right) \left( \begin{array}{l} A_{1j} \\ A_{2j} \end{array} \right) = \left( \begin{array}{c} \delta_{1j} / 2 \pi i \\ -\delta_{2j} / 2 \pi i \\ 0 \\ 0 \end{array} \right)
\]

(8)

The traction integral equation for a source point \( \xi \) with unit normal \( \mathbf{N} \) is obtained by differentiation of Eq. (1) with respect to \( \xi_k \) and the subsequent application of Hooke’s law, to yield

\[
c_{ij}(\xi_m^*) p_j(\xi_m^*) + N_j \int_r s_k^{ij}(\xi_m^*, \xi_n^*) u_j(\xi_n^*) d\Gamma(\xi_m^*) = N_j 
\]

\[
\int_r d_k^{ij}(\xi_m^*, \xi_n^*) p_k(\xi_n^*) d\Gamma(\xi_m^*)
\]

(9)

where \( s_{ijk} \) and \( d_{ijk} \) are thus obtained by differentiation of \( \nu^i_j \) and \( u^i_j \), respectively, with the following expressions

\[
d_k^{ij}(\xi_m^*, \xi_n^*) = -2 \text{Re} \left\{ \sum_{m=1}^2 Q_{jm} R_{jm} \frac{A_{km}}{z_m^* - z_n^*} \right\}
\]

(10)

\[
\left\{ \begin{array}{l} \tilde{s}_{11} \\ \tilde{s}_{22} \end{array} \right\} = \left[ \begin{array}{cc} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{array} \right] \left\{ \begin{array}{l} p_{11}^* \\ p_{22}^* \\ p_{12}^* + p_{21}^* \end{array} \right\}
\]

(11)

where \( C_{ij} \) are the elastic constants of the anisotropic media and

\[
p_{ij,k}(\xi_m^*, \xi_n^*) = -2 \text{Re} \left\{ \sum_{m=1}^2 R_{ik} Q_{jm}(\mu_k n_1 - n_2) \frac{A_{km}}{(z_m^* - z_n^*)^2} \right\}
\]

(12)

\[
R_{jk} = \mu_k \delta_{1j} - \delta_{2j}
\]

(13)

In some mixed BEM approaches the traction BIE (9) is applied at nodes on one side of the crack surface \( \Gamma_+ \) and the standard displacement BIE (1) at nodes on the other side of
the crack $\Gamma_-$ and the rest of the boundary $\Gamma_C$ (Fig. 1). However, if the sum of the tractions on both sides of the crack surface is zero ($\Delta p_j = p_j^+ + p_j^- = 0$), as it happens in most practical applications, it is no longer necessary to apply both equations on the crack surface, since the traction BIE is enough to obtain the COD ($\Delta u_j = u_j^+ - u_j^-$), which is the relevant magnitude in fracture problems.

Therefore, the displacement BIE applied in $\Gamma_C$

$$c_{ij} u_j + \int_{\Gamma_C} p_i^- s_{ij}^k u_j \, d\Gamma + \int_{\Gamma_C} p_i^+ \Delta u_j \, d\Gamma = \int_{\Gamma_C} u_j^0 p_j^+ \, d\Gamma$$

(14)

and the traction BIE applied in $\Gamma_+$

$$p_j + N_j \int_{\Gamma_C} s^+_j u_k \, d\Gamma + N_j \int_{\Gamma_C} s^-_j \Delta u_k \, d\Gamma$$

$$= N_j \int_{\Gamma_C} d^-_{jk} p_k \, d\Gamma$$

(15)

provide a complete set of equations to determine the tractions and displacements on the boundary and the COD along the crack. In Eqs. (14) and (15) $\int$ and $=$ stand for the Cauchy Principal Value and the Finite Part of the integral, respectively. Note that the difference between Eq. (15) and the general traction BIE (9) is the free term, which is set to 1 to account for the additional singularity arising from the coincidence of the two crack surfaces.

The discretization approach follows Sáez et al. [15] and it is summarized in Fig. 1. Thus, in order to fulfill the $C^1$ continuity of the displacements at collocation points that limit to the boundary requires, non-conforming (discontinuous) quadratic elements are considered to mesh the crack (Fig. 2). To reproduce adequately the displacements behavior near the crack tip, simple straight line discontinuous quadratic elements with the mid-node located at one quarter of the element length (Fig. 3) are used. Standard continuous quadratic elements are used for the rest of the boundary, except for the case when there is an intersection between a crack and an external boundary. In such a case, a semi-discontinuous element is used on the outer boundary to avoid a common node at the intersection. The geometry of the discontinuous quadratic elements is represented using the same shape functions $\phi_{iG}$, $\phi_{fG}$ and $\phi_{NG}$ of the standard (continuous) quadratic elements (Fig. 4(a)), whilst the boundary variables $u_k$, $p_k$ and $\Delta u_k$ are represented in terms of their values at the collocation nodes NC1, NC2 and NC3 and the shape functions $\phi_1$, $\phi_2$ and $\phi_3$ (Fig. 4(b)).

3. Treatment of singular integrals

Integrals of the kernels $u_j$ in Eq. (14) show a weak singularity of the type $0[\ln(x_j^+ - x_j^-)]$ as $x \rightarrow \xi$, which can be numerically integrated without difficulty by using a special quadrature including a logarithm.

The kernels $p_i^+$ and $d_{ik}$ appearing in Eqs. (14) and (15) show a strong singularity of $0[1/(x_j^+ - x_j^-)]$ as $x \rightarrow \xi$. The integration of such kernels over the element $\Gamma_i$ that contains the collocation point $\xi$ can be done as follows:

After discretization, integrals of the type

$$\int_{\Gamma_i} p_i^+ \phi_j \, d\Gamma$$

(16)

where $\phi_j$ are the BE shape functions, contain basic singular integral as (see Eq. (5))

$$I_k = \int_{\Gamma_i} (\mu_k n_1 - n_2) \frac{1}{\xi_k - \xi_{k'}} \phi_k \, d\Gamma, \quad k = 1, 2$$

(17)

Calling

$$r_k = \xi_k - \xi_{k'} = (x_1 - \xi_1) + \mu_k (x_2 - \xi_2), \quad k = 1, 2$$

(18)

it follows that

$$\frac{dr_k}{d\Gamma} = \frac{dr_k}{dx_1} \frac{dx_1}{d\Gamma} + \frac{dr_k}{dx_2} \frac{dx_2}{d\Gamma} = -n_2 + \mu_k n_1$$

(19)

where $n_1$, $n_2$ are the components of the external unit normal to the boundary at the observation point $x$ (Fig. 5). Note that the variable $r_k$ defining the relative location of the collocation and observation points is complex. Eq. (19) is the key idea for all the transformations proposed below and it shows that the Jacobian $dr_k/d\Gamma$ of the coordinate
transformation that maps the geometry of the BE G onto the complex plane r_k is included in the fundamental solution itself (see Eqs. (5) and (12)). This fact will permit to simplify notably the approach proposed in Ref. [15] for isotropic media, resulting in much simpler expressions for the anisotropic case.

Taking into account Eq. (19), Eq. (17) can be rewritten as

\[ I_k = \int_{r_k} \frac{1}{r_k} \phi_q \, dr_k \]

\[ \text{which can be decomposed into the sum of a regular integral plus a singular integral with known analytical solution} \]

\[ I_k = \int_{r_k} \frac{1}{r_k} (\phi_q - 1) \, dr_k + \int_{r_k} \frac{1}{r_k} \, dr_k \]

Alternatively, the singular integration in Eq. (16) and the computation of the \( c_{ij}(\xi) \) terms in Eq. (14) can be avoided by using the rigid body motion condition.

The integration of the kernels \( d_{ijk}N_j = d_{ik}^s \) can be done in a similar way as for the \( p_{ik}^s \) kernels since they contain singularities of the same type when \( x \to \xi \). Though such singular integration is not needed when the sum of the tractions on both sides of the crack surface is zero (Eq. (15)), it is included here for the sake of completeness.

From Eqs. (7) and (10) it follows that the singular integrals are of the type

\[ I_k' = \int_{r_k} (\mu_k n_1 - n_2) \frac{1}{r_k} \phi_q \, d\Gamma, \quad k = 1, 2 \]

which can be regularized as follows

\[ I_k' = \int_{r_k} \frac{1}{r_k} \left( \mu_k n_1 - N_2 \right) \frac{dr_k}{d\Gamma} \phi_q \, d\Gamma + \int_{r_k} \frac{1}{r_k} \phi_q \, dr_k \]

The first integral in Eq. (23) is regular, since \( n_l \to N_j \) when \( x \to \xi \) and Eq. (19) holds. The second integral has been previously studied (Eqs. (20) and (21)).

4. Treatment of hypersingular integrals

The integration of the \( s_{ijk}^s \) kernels in Eq. (15) leads to hypersingular integrals since they show a singularity of the type \( 0/[\mu_k (z_1 - z_2)^2] \) as \( x \to \xi \). From Eqs. (11) and (12) it follows that the hypersingular integrals are of the form

\[ I_k'' = \int_{r_k} (\mu_k n_1 - n_2) \frac{1}{r_k} \phi_q \, d\Gamma = \int_{r_k} \frac{1}{r_k} \phi_q \, dr_k, \]

where Eqs. (18) and (19) have been taken into account. The integral \( I_k'' \) can again be decomposed into the sum of a regular integral plus singular integrals with known analytic solution by using Eq. (19) and the first two terms of the series expansion of the shape function \( \phi_q \) at the collocation point

\[ \phi_q(r_k) = \phi_q(r_k = 0) + \frac{d\phi_q}{dr_k} \bigg|_{r_k = 0} r_k + 0(r_k^2) \]
Thus, $I^0_k$ can be written as

$$I^0_k = \int_{r_k} \frac{1}{r_k^2} \phi_k r_k \, dr_k = \int_{r_k} \frac{1}{r_k^2} (\phi_k - \phi_k(0) + \phi_k(0)/r_k) \, dr_k$$

$$+ \phi_k(0) \int_{r_k} \frac{1}{r_k} \, dr_k + \phi_k(0) \int_{r_k} \frac{1}{r_k} \, dr_k$$

(26)

The first integral in Eq. (26) is regular and the other two can be easily computed analytically.

Since the series expansion is considered for the shape functions, which have simple expressions, the procedure presented is more general and easier to implement than the one given in Ref. [10], where the expansion is done for a more complicated function $f(\zeta)$ defined by the product of the anisotropic fundamental solution, a shape function, the Jacobian $J(\zeta)$ of the coordinate transformation to the local parametric coordinate $\zeta$ and the hypersingular term $(\zeta - \zeta_0)^2$. $\zeta_0$ defines the collocation point location. In Ref. [10], all the integrals are solved analytically for straight elements, since for that case the Jacobian is constant. The generalization of such procedure to curved elements is rather complicated due to the complexity of the kernel function $f(\zeta)$.

Pan and Amadei [12] evaluated the hypersingular integrals numerically by using one of the existing quadrature rules [13]. However, the existence of different quadratures for hypersingular integrals (see, e.g. Kutt [17] or Ladopoulos [18] among others) shows that the numerical evaluation of such integrals is still an open issue, where efficiency and accuracy have to be compromised.

The approach developed in this paper permits the simple integration of all the kernels appearing in both displacement and traction BIE for any curved or straight quadratic element.

5. Crack modelling and SIF evaluation

The generality of the integration method developed in previous sections allows for the use of straight line quarter-point quadratic elements with collocation points located extremely close to the crack tip (Fig. 3), as previously presented in Ref. [15] for isotropic materials. By doing so, the $\sqrt{r}$ behavior that the COD shows in the vicinity of the crack front as the distance to the crack front $r$ tends to zero [19] is reproduced, provided that for such geometry the relation between $r$ and the local dimensionless coordinate $s$ is given by

$$s = 2\sqrt{\frac{r}{L}} - 1$$

(27)

In the present work, the collocation nodes NC1, NC2 and NC3 are located at $s_1 = -3/4$, $s_2 = 0$, and $s_3 = +3/4$, respectively (Fig. 6). In such case, the distance $r$ from the collocation nodes of the quarter-point element to the crack tip follows from Eq. (27).

$$r_1 = \frac{L}{64} \text{ for NC1}$$

$$r_2 = \frac{L}{4} \text{ for NC2}$$

$$r_3 = \frac{49L}{64} \text{ for NC3}$$

(28)

Defining the usual polar coordinate system $\hat{r} - \theta$, centered at the crack tip and such that $\theta = \pm \pi$ are the crack surfaces, the normal $(\Delta u_2)$ and shear $(\Delta u_1)$ components of the COD along the crack surfaces for a plane stress situation can be written as (from Ref. [19])

$$\begin{align*}
\{ \Delta u_1 &= u_1^+ - u_1^- \\
\Delta u_2 &= u_2^+ - u_2^- \}
= \sqrt{\frac{2r}{\pi}} \left( \begin{array}{c} D_{11} \\
D_{21} \\
D_{22} \end{array} \right) \left( \begin{array}{c} K_I \\
K_{II} \end{array} \right)
\end{align*}$$

(29)

according to Linear Elastic Fracture Mechanics. Only the leading term of the asymptotic behavior of the COD near the crack tip has been considered in Eq. (29). The superscripts + and _ denote the two different crack faces, $K_I$ and $K_{II}$ are the SIFs for the modes I and II, respectively, and

$$D_{11} = \text{Im} \left( \frac{\mu_2 P_{11} - \mu_1 P_{12}}{\mu_1 - \mu_2} \right)$$

$$D_{21} = \text{Im} \left( \frac{\mu_2 P_{21} - \mu_1 P_{22}}{\mu_1 - \mu_2} \right)$$

(30)

$$D_{12} = \text{Im} \left( \frac{P_{11} - P_{12}}{\mu_1 - \mu_2} \right)$$

$$D_{22} = \text{Im} \left( \frac{P_{21} - P_{22}}{\mu_1 - \mu_2} \right)$$

Particularizing Eq. (29) for $r = L/64$ (collocation node NC1), the following one point displacement formula for

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{Discontinuous quarter-point element with collocation nodes at $s_1 = -3/4$; $s_2 = 0$; $s_3 = 3/4$.}
\end{figure}
the direct evaluation of the SIF can be obtained

$$
\begin{bmatrix}
K_I \\
K_{II}
\end{bmatrix} = \frac{4\sqrt{2\pi/L}}{D_{11}D_{22} - D_{12}D_{21}} \left( \begin{array}{cc} D_{22} & -D_{12} \\
-D_{21} & D_{11} \end{array} \right) \left( \begin{array}{c} \Delta u^1_{NC1} \\
\Delta u^2_{NC1} \end{array} \right)
$$

Several other approaches to evaluate the SIF directly from the computed nodal values may be considered, as described in Ref. [15] for the isotropic case. However, the procedure presented in Eq. (31) is the more accurate and the less mesh-dependent one, since the SIF is calculated from the COD at a collocation point extremely close to the crack front, where the \( \sqrt{r} \) behavior of the COD is highly dominant. As compared to the J-integral technique, Eq. (31) provides the SIF directly from the nodal displacement values with no need of postprocessing.

It should be pointed out that the use of this quarter-point element does not need on any additional effort for its implementation.

6. Numerical examples

The use of hypersingular discontinuous quarter-point elements is illustrated next. Several examples with different geometries and material properties are considered to show the generality of the present method and its accuracy for simple meshes.

6.1. Griffith crack problem

The first example corresponds to a straight crack under internal pressure in an unbounded plane. This simple problem has been taken as a benchmark since it has known analytical solution. Only one side of the crack surface has to be discretized. Six equal length quadratic elements are used in the present case with the two extreme ones being quarter-point. The problem was analysed for many different material properties, isotropic and anisotropic with a shear modulus \( G_{12} = 6 \text{ GPa} \) and the Poisson’s ratio \( \nu_{12} = 0.03 \) whilst the Young’s moduli are determined as a function of the parameter \( \varphi = E_1/E_2 \) as \( E_1 = G_{12}(\varphi + 2\nu_{12} + 1) \) and \( E_2 = E_1/\varphi \). Values of \( \varphi \) between \( 10^{-3} \) and \( 10^3 \) were considered. The obtained mode I SIF values are non-dependent on the elastic properties and in all cases within 0.1% of the exact solution.

6.2. Central crack in orthotropic plate

The second example corresponds to a square plate \( (h/w = 1) \) with a central crack of length \( 2a \). The plate is subjected to a uniform traction at two opposite sides, as shown in Fig. 7. Different material properties are studied as in the previous example, so that the shear modulus \( G_{12} = 6 \text{ GPa} \) and the Poisson’s ratio \( \nu_{12} = 0.03 \) have fixed values, \( \varphi = E_1/E_2 \) as \( E_1 = G_{12}(\varphi + 2\nu_{12} + 1) \) and \( E_2 = E_1/\varphi \).

The BE discretization consists of 24 elements on the external boundary and six elements on the crack.

Results for the normalized mode I SIF \( (K_I/\sqrt{\pi a}) \) are given in Table 1 for the parameter \( \varphi \) varying from 0.1 to 4.5 and a crack length \( a = 0.2w \). Our results are compared with those obtained by Bowie and Freeze [20] using a modified mapping collocation technique, and Sollero [9] and Sollero and Aliabadi [10] using the dual BEM together with the J-integral to compute the SIF. Table 2 contains the results for a crack length \( a = 0.5w \). The present results show an excellent agreement with the other two solutions.

6.3. Double edge crack in anisotropic plate

This example corresponds to a double edge crack in a square plate \( (h/w = 1) \) subjected to a uniform traction at two
opposite ends, as shown in Fig. 8. The plate is a symmetric angle ply composite laminate consisting of four graphite-epoxy laminae with the following elastic constants:

\[ E_1 = 144.8 \text{ GPa, } E_2 = 11.7 \text{ GPa, } G_{12} = 9.66 \text{ GPa and } \nu_{12} = 0.21. \]

The directions of the fibers are rotated from \( \phi = 0 \) to \( \phi = 90^\circ \).

Since the laminate is symmetric, normal and shear strains are uncoupled and due to the symmetry of the problem, only half of the plate is discretized. The BE mesh consists of 24 elements on the external boundary and six elements on the crack.

Results for the normalized mode I SIF are given in Table 3 for a crack length \( a = 0.5w \). The computed results are compared to those obtained by Chu and Hong [21] using the finite element method, and Sollero [9] using the dual BEM. The \( J \)-integral technique was used in both references to compute the SIF. The present results show again an excellent agreement with both the other solutions.

### 6.4. Central slant crack in anisotropic plate

This example corresponds to a rectangular plate with an inclined central crack as shown in Fig. 9. The plate is subjected to a uniform traction at two opposite sides. The height to width ratio of the plate is \( h/w = 2 \). The crack has a length \( 2a = 0.4w \) and it is inclined \( 45^\circ \) with respect to the plate sides. The material is a glass-epoxy composite with properties: \( E_1 = 48.26 \text{ GPa, } E_2 = 17.24 \text{ GPa, } G_{12} = 6.89 \text{ GPa and } \nu_{12} = 0.29. \)

The direction of the fibers is rotated from \( \phi = 0 \) to \( 180^\circ \).
The BE discretization consists of six discontinuous quadratic elements on the crack surface and 24 quadratic isoparametric elements on the external boundary.

The computed values of normalized mode I and mode II SIF are shown in Tables 4 and 5, respectively. They are compared with those obtained by Gandhi [22] using a mapping collocation method, and Sollero and Aliabadi [10] using the dual BEM together with the $J$-integral technique. Again, excellent agreement is observed.

6.5. Double crack emanating from a hole in anisotropic plate

For the last example, a rectangular plate with two symmetric cracks emanating from a circular hole is considered, as shown in Fig. 10. The diameter to width ratio is $r/w = 0.5$ and the crack length to width ratio $a/w$ range from 0.55 to 0.70. The material is an unidirectional boron-epoxy composite laminate with elastic constants: $E_1 = 204$ GPa, $E_2 = 18.5$ GPa, $G_{12} = 5.59$ GPa and $\nu_{12} = 0.23$. The SIF are computed for different fiber orientations ranging from $\phi = 0$ to 90°. The plate is subjected to a uniform tensile stress at its ends.

The BE mesh consists of 34 elements on the external boundary, 28 elements on the hole and five elements on each crack surface.

Normalized mode I and mode II SIF are presented in Fig. 11. The computed results are compared with those obtained by Sollero et al. [23] using the BEM subregion technique and the $J$-integral. A good agreement between both solutions is observed. As expected, the mode II SIF is zero for $\phi = 0$ and 90° since for that fiber orientation there is no coupling between the normal and shear deformations for the considered loading conditions. It is worth to mention that the current results show almost no dependency on the discretization and the differences between them and those in Ref. [23] (also shown in Fig. 11) are most probably due to the fact that values obtained with a subregion technique and with no singular elements for the internal boundaries are less accurate than those obtained with a single domain hypersingular formulation as the present one. This fact was also pointed out by Sollero and Aliabadi [10] for isotropic crack problems. No other results obtained using a hypersingular or dual formulation for this anisotropic problem exist in the literature.

6.6. Curved cracks in unbounded domain

In order to show the use of the current procedure for curved crack geometries, the problem of a crack with circular arch shape in an unbounded domain is analysed next. The crack region is under uniform traction in two perpendicular directions. Two different materials are considered, one is isotropic and the other is a graphite-epoxy laminae with elastic constants: $E_1 = 138.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = 7.1$ GPa and $\nu_{12} = 0.3$. Cracks with semi-angles between 15 and 75° are considered. The BE mesh for any crack angle contains 10 elements as shown in Fig. 12, where dots indicate extreme points.
of the BEs. The isotropic material problems were studied using eight curved quadratic elements and two very small quarter-point straight elements at the tips (Fig. 12). The computed mode I and II SIF values are shown in Fig. 13. They are compared with the exact analytical solution obtained by Muskhelishvili [24] for the same problem. The agreement between the two sets of results is excellent.

Once the accuracy of the procedure for curved cracks has been tested, the effect of a good representation of the crack curvature is studied. To do so, the same crack, but in the graphite-epoxy material, is analysed using the same curved element mesh and also using a mesh of ten straight line quadratic elements with the extreme nodes at the same position as in the curved element case. The difference between the curved elements results and those obtained with straight line elements are shown in Fig. 14. It can be seen that the difference increases with the crack angle and it gets up to a 4.1% for a 75 semi-angle. These results show the convenience of using curved elements to represent curved cracks unless the price of using a very large number of elements is paid.
7. Conclusions

In this paper, the mixed BE formulation using displacement and traction integral equations for 2D anisotropic media is used and a general approach for the evaluation of the strongly singular and hypersingular integrals for general curved BEs is presented. Such integrals are analytically transformed into regular ones by the simple election of an integration variable which is consistent with that of the anisotropic fundamental solution.

The procedure is completely general and allows for the use of curved or straight elements along the crack or any external boundary. Furthermore, this very simple integration procedure can be easily extended to other 2D BE formulations for anisotropic materials with fundamental solutions obtained by means of the complex variable function method, like those involving half-plane, bimaterial or piezoelectric problems, as it will be illustrated in a forthcoming paper.

In particular, the generality of the integration method allows for the use of straight line quarter-point quadratic elements and collocation points located extremely close to the crack tip (as in Ref. [15]). By doing so, the behavior of the COD is reproduced and the SIF can be evaluated very accurately by direct use of the COD at points where this behavior is highly dominant. Similarly, this simple and accurate SIF computation approach can be extended to dynamic crack problems in the context of a frequency domain or time domain mixed formulation.

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References