Lorentzian manifolds with no null conjugate points

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(Received 27 November 2002; revised 2 April 2003)

Abstract

An integral inequality for a compact Lorentzian manifold which admits a timelike conformal vector field and has no conjugate points along its null geodesics is given. Moreover, equality holds if and only if the manifold has nonpositive constant sectional curvature. The inequality can be improved if the timelike vector field is assumed to be Killing and, in this case, the equality characterizes (up to a finite covering) flat Lorentzian $n(\geq 3)$-dimensional tori. As an indirect application of our technique, it is proved that a Lorentzian 2–torus with no conjugate points along its timelike geodesics and admitting a timelike Killing vector field must be flat.

1. Introduction

Conjugate points on geodesics of a Lorentzian manifold have been systematically studied [3, chapters 10, 11], [18, chapter 10]. Of course, their definition in Lorentzian geometry is formally the same as the Riemannian case, but there are here several geometric behaviours which are not possible or have no sense for a Riemannian manifold. In fact, on a Lorentzian manifold the geodesics are divided into three classes, spacelike, null and timelike, according to their causal character, and consequently conjugate points are classified into spacelike, null and timelike points (the last two cases, i.e. causal conjugate points, have been extensively considered because of their physical meaning). Let us remark that there are also different approaches and techniques to deal with timelike or null conjugate points. With respect to this class of conjugate points, let us recall that no null geodesic in a 2-dimensional Lorentzian manifold has conjugate points [3, lemma 10-45] and on the other hand, there is a

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§ Partially supported by MCYT-FEDER Grant BFM2001-2871-C04-01.
simple but surprising fact which asserts that there are no null conjugate points (i.e. conjugate points on null geodesics) in a Lorentzian manifold of constant sectional curvature. The converse obviously is not true; in fact, it is easy to construct a Lorentzian manifold (even geodesically complete) with no null conjugate points and which does not have constant sectional curvature. The following question arises then in a natural way:

*Under what assumption does a Lorentzian manifold with no conjugate points along its null geodesics have constant sectional curvature?*

The main aim of this paper is to find a satisfactory answer to this question. In order to face it we will introduce, following the technique developed in \[11\], an integral inequality on a remarkable family of Lorentzian manifolds such that no null geodesic contains a pair of mutually conjugate points. Later we will use it to get the desired answer.

Our efforts to deal with this problem have been inspired by several important contributions in Riemannian geometry. Following the philosophy to extend properties of Riemannian manifolds of nonpositive sectional curvature to Riemannian manifolds with no conjugate points, E. Hopf [13] proved in 1948 that:

*The total scalar curvature of a closed surface with no conjugate points is nonpositive and vanishes only if the surface is flat.*

Therefore, this result combined with the Gauss–Bonnet theorem gives the following relevant fact:

*A Riemannian torus with no conjugate points must be flat.*

Note that the same conclusion trivially holds if the Riemannian metric on the torus is assumed to have nonpositive Gauss curvature. The Hopf theorem was extended by L. W. Green in 1958 [9] to any dimension to obtain:

*If a compact Riemannian manifold \((B, g)\) has no conjugate points, then its scalar curvature \(S\) satisfies*

\[
\int_B S \, d\mu_g \leq 0,
\]

*and equality holds if and only if the metric is flat.*

More recently, F. Guimaraes [10] has generalized the Green theorem to complete Riemannian manifolds under the assumption that the Ricci curvature has an integrable positive or negative part on the unit tangent bundle.

Let us now recall that the no conjugacy assumption on a fixed geodesic in a Riemannian manifold or a causal geodesic in a Lorentzian manifold has been used by Ehrlich and Kim in [7, theorem 3-1] to obtain a generalization of the Hawking-Penrose conjugacy theorem (see [3, theorem 12-18], for instance) of Singularity theory.

In this paper we will consider two assumptions on Lorentzian manifolds to give an answer to the previous question: compactness and the existence of certain symmetry of the metric tensor defined by a timelike conformal vector field. The first requirement is due to technical reasons (we will use integration on the manifold) and that we set our approach into global geometry. Of course, a compact Lorentzian
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A compact Lorentzian manifold which admits a timelike conformal vector field must be geodesically complete.

Let us remark that the study of conformal vector fields (with certain causal character) on Lorentzian manifolds is a topic of special importance. It has been developed mainly under assumptions of interest in Physics. On the other hand, timelike conformal (and more general) vector fields have shown to be manageable for the use of a suitable adaptation of the Bochner technique on compact Lorentzian manifolds [19, 21, 22]. This has been fruitful to get several obstructions and classifications results.

The particular case of timelike Killing vector fields was previously considered as a useful tool in this approach [1, 14, 17, 23, 24] and references therein.

The main results of this paper are Theorem 4.1 and Corollary 4.3 and can be summarized as follows:

If $(M, g)$ is an $n(\geq 3)$-dimensional compact Lorentzian manifold with no null conjugate points and which admits a timelike conformal vector field $K$, then

$$\int_M [n \text{Ric}(U, U) + S] h^n d\mu_g \leq 0$$

and equality holds if and only if $(M, g)$ has constant sectional curvature $k \leq 0$.

Here Ric and $S$ are, respectively the Ricci tensor and the scalar curvature of $(M, g)$, $h$ is the function $1/\sqrt{-g(K, K)}$ and $U$ is the unit timelike vector field $hK$.

The difference with the Riemannian case should be pointed out, in which it does not need the existence of a conformal vector field but it needs the absence of conjugate points along all geodesics to get global information on the manifold.

If the conformal vector field $K$ is assumed to be a Killing vector field this result can be improved, Theorem 4.4, to get

If $(M, g)$ is an $n(\geq 3)$-dimensional compact Lorentzian manifold with no null conjugate points and which admits a timelike Killing vector field $K$, then

$$\int_M S h^n d\mu_g \leq 0$$

and equality holds if and only if $(M, g)$ is isometric (up to a finite covering) to a flat Lorentzian $n$-torus. In particular, in this case $K$ is parallel, the first Betti number of $M$ is not zero and the Levi–Civita connection of $g$ is Riemannian.

This result widely extends a theorem of Kamishima. In fact, he proved in [14, theorem A] that if a compact Lorentzian manifold with constant sectional curvature $k \in \mathbb{R}$ admits a timelike Killing vector field, then $k \leq 0$. Moreover, if $k = 0$, then the manifold is affinely diffeomorphic to a Riemannian manifold with non zero first Betti number. It should be noted that Kamishima uses a very different technique in [14], strongly depending on the Lie group machinery of Lorentzian space forms, than the present one. Moreover, Theorem 4.4 permits to reprove in Corollary 4.6 the previously mentioned Hopf–Green inequality in Riemannian Geometry [9].

Although we deal here with null conjugate points on $n(\geq 3)$-dimensional Lorentzian manifolds, by the use of a trick we are able to prove, Corollary 4.8, that:
A compact Lorentzian surface admitting a timelike Killing vector field with no conjugate points along its timelike geodesics must be flat.

This result resembles the Hopf theorem [13], but in our case the no conjugacy hypothesis involves only timelike geodesics. It should be noted that a Lorentzian torus which admits a timelike Killing vector field is conformally flat [23]. Theorem 4-8 gives an answer to the natural question to decide when a Lorentzian torus with a timelike Killing vector field must be flat. Finally, in the last section we illustrate our results with remarkable examples of compact Lorentzian manifolds.

2. Preliminaries

Let \((M, g)\) be an \(n(\geq 2)\)-dimensional Lorentzian manifold; that is a (connected) smooth manifold \(M\) endowed with a non-degenerate metric \(g\) with signature \((- + , \ldots , + )\). We shall write \(\nabla\) for the Levi–Civita connection of \(g\), \(R\) for its Riemannian curvature tensor (our convention on the curvature tensor is \(R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z\)), Ric for its Ricci tensor, \(S\) for its scalar curvature and \(d\mu\) for the canonical measure associated with \(g\).

As usual, the causal character of a tangent vector \(v \in T_pM\) is timelike (resp. null, spacelike) if \(g(v, v) < 0\) (resp. \(g(v, v) = 0\) and \(v \neq 0\), \(g(v, v) > 0\) or \(v = 0\)). If \(v \in T_pM\), then \(\gamma_v\) will denote the unique (maximal) geodesic which satisfies \(\gamma_v(0) = v\). The causal character of \(\gamma_v(t)\) is independent of \(t\) and it is called the causal character of the geodesic. A null geodesic \(\gamma_v\) of \((M, g)\) is then a geodesic such that \(v\) is a null vector. From now on, we will write \(P^b\) for the parallel translation along \(\gamma_v\) from \(\gamma_v(a)\) to \(\gamma_v(b)\).

Unless we indicate otherwise, \((M, g)\) will denote an \(n(\geq 3)\)-dimensional Lorentzian manifold which admits a timelike vector field \(K\). We put \(h = 1/\sqrt{-g(K, K)}\) and write \(U = hK\) for the normalized vector field obtained from \(K\).

Recall that the null congruence on \(M\) associated to \(K\) is the subset of the tangent bundle \(TM\) given by

\[C_K M = \{ v \in TM : g(v, v) = 0 \text{ and } g(v, K_{\pi v}) = 1 \},\]

where \(\pi : TM \to M\) is the natural projection [12, 16].

This subset is an orientable embedded submanifold of \(TM\) with dimension \(2(n - 1)\), and can be naturally endowed with the Lorentzian metric \(\hat{g}\) obtained by restriction of the Sasaki metric \(g\) of \(TM\) (note that \(g\) is non-degenerate with signature \((- - + , \ldots , + )\)). Moreover, now \(\pi : (C_K M, \hat{g}) \to (M, g)\) is a semi-Riemannian submersion with spacelike fibers. Let us suppose that the vector field \(K\) is conformal (i.e. \(\hat{g} = \rho g\) with \(\rho \in C^\infty(M)\)). Under that assumption we have that \(C_K M\) is invariant by any (local) flow of the geodesic vector field. Furthermore, if \(M\) is also assumed to be compact, then it must be geodesically complete [20], and moreover, in this case, we have

\[\int_{C_K M} (f \circ \Phi_t) d\mu_{\hat{g}} = \int_{C_K M} f \ d\mu_{\hat{g}}, \quad (2.1)\]

for all \(f \in C^0(M)\) and \(t \in \mathbb{R}\) where \(\Phi_t(v) = \gamma_v(t)\) is the geodesic flow. (See details on those properties of \(C_K M\) in [11].)

As a key tool to get our main results in this paper we will use [11, theorem 3-2] which reads as follows: Let \((M, g)\) be an \(n(\geq 3)\)-dimensional compact Lorentzian
manifold that admits a timelike conformal vector field $K$. If there is a $> 0$ such that every null geodesic $γ_v$, with $v \in C_K M$, has no conjugate point of $γ_v(0)$ in $[0, a)$, then

$$\text{Vol}(C_K M) \geq \frac{a^2}{\pi^2(n-2)} \int_{C_K M} \tilde{\text{Ric}} \, dμ_3,$$

where $\tilde{\text{Ric}}$ denotes the quadratic form associated with the Ricci tensor. Moreover, equality in (2.2) has been characterized by using null sectional curvature.

In order to fix the notation to be used later, we recall the well-known notion of warped product (see [18, definition 7.33], for instance). Let $(B, g_B)$ and $(F, g_F)$ be semi-Riemannian manifolds. We write $π_B$ and $π_F$ for the projections of $B \times F$ onto $B$ and $F$, respectively. Let $f > 0$ a smooth function on $B$. The warped product $B \times_F F$ is the product manifold $B \times F$ furnished with the metric tensor $g_f = π_B^*(g_B) + (f \circ π_B)^2 π_F^*(g_F)$.

### 3. Null geodesics with no conjugate points

It is well known that if $γ_v : [0, a] \to M$ is a null geodesic such that there is no conjugate point of $γ_v(0)$ in $[0, a)$, then the Hessian form $H_{γ_v}^⊥$ is positive semidefinite, i.e. it satisfies

$$H_{γ_v}^⊥(V, V) = \int_0^a \left[ g \left( \frac{\nabla V}{dt}, \frac{\nabla V}{dt} \right) - g(R(γ_v, γ_v'), γ_v', V) \right] dt \geq 0,$$

for every piecewise smooth vector field $V$ along $γ_v$ such that $V(0) = 0, V(a) = 0$ and $g(γ_v, V) = 0$ [18, p. 290-1].

The following construction, similar to the one made in [3, chapter 10], emerges in a natural way in our setting. For every null tangent vector $v \in T_p M$, we consider the quotient vector space $N(v) = \langle v \rangle^⊥ / \langle v \rangle$, where $\langle v \rangle$ denotes the vector subspace of $T_p M$ spanned by $v$. If $x \in \langle v \rangle^⊥$, then $[x]$ will denote the element of $N(v)$ defined by its equivalence class. Next, we will project on $N(v)$ several geometric objects to be used later. Given $[x], [y] \in N(v)$, if we put $\overline{g}([x], [y]) = g(x, y)$, then $\overline{g}$ is a (well-defined) positive definite scalar product on $N(v)$. The curvature operator may also be projected on $N(v)$ as $\overline{R}([x], [v]) = [R(x, v)v]$. Finally, since $P_0^t(v) = γ_v(t)$ we can define $\overline{P}_0^t(x) = [P_0^t(x)] \in N(γ_v(t))$.

To meet our needs, $γ_v(t)$ is considered to be defined for all $t \in \mathbb{R}$. Let $\mathcal{X}(γ_v)$ be the $C^∞(\mathbb{R})$—module of vector fields along $γ_v$ and $\mathcal{X}^⊥(γ_v) = \{ V \in \mathcal{X}(γ_v) : g(V, γ_v') = 0 \}$, which is clearly a submodule of $\mathcal{X}(γ_v)$. If $[V] \in \mathcal{X}^⊥(γ_v)/γ_v'$, where $γ_v'$ denotes the submodules of $\mathcal{X}^⊥(γ_v)$ spanned by $γ_v$, then we can define $[V](t)$ to be $[V(t)] \in N(γ_v(t))$. Now, it is natural to put $\nabla^2 [V] = [\nabla V]$. Finally, if $[V](0) = [0], [V](a) = [0]$, we set $\overline{H}_{γ_v}^⊥([V], [V]) = H_{γ_v}^⊥(V, V)$ where $V \in [V]$ with $V(0) = 0$ and $V(a) = 0$.

A class $[V]$ is said to be a Jacobi class along $γ_v$ if

$$\nabla^2 [V] + \overline{R}([V], γ_v')γ_v' = [0].$$

It is not difficult to show that $γ_v(0)$ and $γ_v(a)$, with $a \neq 0$, are conjugate along $γ_v$ if and only if there exists a Jacobi class $[V](\neq [0])$ along $γ_v$ which satisfies $[V](0) = [0]$ and $[V](a) = [0]$. In fact, any representant $V \in \mathcal{X}^⊥(γ_v)$ of such a Jacobi class must
satisfy
\[ \frac{\nabla^2 V}{dt^2} + R(V, \gamma'_v)\gamma'_v = f\gamma'_v, \]
where \( V(0) = \tau_1 v, \) \( V(a) = \tau_2 \gamma'_v(a), \) with \( \tau_1, \tau_2 \in \mathbb{R}, \) and \( f \) is a smooth function. In this case, if we choose a function \( F \) such that \( d^2 F/ dt^2 = f, \) then
\[
J(t) = V(t) + \left[ \frac{(a-t)(F(0) - \tau_1) + t(F(a) - \tau_2)}{a} - F(t) \right] \gamma'_v(t)
\]
is a nonzero Jacobi vector field along \( \gamma_v, \) with \( J(0) = 0 \) and \( J(a) = 0. \)

Several properties of Jacobi classes below are similar to the well-known ones of the Jacobi vector fields; for instance [18, lemmas 10-15, 10-16] give respectively:

**Lemma 3.1.**
1. For every Jacobi classes \( [V], [W] \) along \( \gamma_v, \) with \( [V](c) = [W](c) = 0 \) for some \( c \in \mathbb{R}, \) we have
\[
\bar{g} \left( \frac{\nabla [V]}{dt}, [W] \right) = \bar{g} \left( [V], \frac{\nabla [W]}{dt} \right).
\]
2. If \([J_1], \ldots, [J_k]\) are Jacobi classes along \( \gamma_v \) such that \([J_i]\) and \([J_j]\) satisfy previous equation (3.2) for all \( i, j \) and we put \( [X] = \sum_{i=1}^{k} \lambda_i [J_i] \in \mathbb{R}^+(\gamma_v) / \langle \gamma_v \rangle, \) \( \) with \([X](0) = [0] \) and \([X](a) = [0]. \) then
\[
\bar{P}_{\gamma_v|_{0,a}}([X], [X]) = \int_{0}^{a} \bar{g} \left( \sum_{i=1}^{k} \frac{d\lambda_i}{dt} [J_i], \sum_{j=1}^{k} \frac{d\lambda_j}{dt} [J_j] \right) dt.
\]

Let \( \text{End}(\mathbb{N}(v)) \) be the space of all \( \mathbb{R} \)-linear operators of \( \mathbb{N}(v) \) and \( \mathcal{R} : \mathbb{R} \to \text{End}(\mathbb{N}(v)) \) the smooth path in \( \text{End}(\mathbb{N}(v)) \) given by \( \mathcal{R}(t) = \bar{P}_v^t \circ \bar{R}(\gamma'_v(t)) \circ \bar{P}_0^t. \)

From the classical symmetries of the curvature tensor \( R \) we get that \( \mathcal{R}(t) \) is self-adjoint with respect to \( \bar{g}. \) One associates to \( \mathcal{R} \) the matrix Jacobi equation
\[
\frac{d^2 \mathcal{X}}{dt^2} + \mathcal{R} \circ \mathcal{X} = 0,
\]
where \( \mathcal{X} : \mathbb{R} \to \text{End}(\mathbb{N}(v)) \) is smooth. One easily checks that if \( \mathcal{X} \) is a solution to (3.4) and \([x] \in \mathbb{N}(v). \) then \( \bar{P}_0^t(\mathcal{X}(t)[x]) \) is a Jacobi class along \( \gamma_v. \)

In all what follows \( \mathcal{A} \) will represent the solution to (3.4) determined from the initial conditions \( \mathcal{A}(0) = 0 \) and \( d\mathcal{A}/dt|_{0} = Id, \) where \( Id \) denotes the identity transformation.

For every \( x \in \langle \gamma_v \rangle', \) let us denote \( J_x \) the unique Jacobi vector field along \( \gamma_v \) such that \( J_x(0) = 0 \) and \( \sum_{i=1}^{k} \frac{d\lambda_i}{dt} |_{0} = x. \) It is not difficult to show that \( \mathcal{A}(t)[x] = \bar{P}_v^t J_x(t). \)

**Lemma 3.2.** Let \( \gamma_v \) be a null geodesic, then there is no conjugate point of \( \gamma_v(0) \) along \( \gamma_v \) if and only if \( \text{Ker} \mathcal{A}(t) = 0 \) for all \( t \neq 0. \)

**Proof.** Let \( J \) be a nonzero Jacobi field along \( \gamma_v \) such that \( J(0) = 0 \) and \( J(a) = 0 \) for \( a \neq 0. \) If \( \sum_{i=1}^{k} \frac{d\lambda_i}{dt} |_{0} = x, \) then \( 0 \neq [x] \in \text{Ker} \mathcal{A}(a). \) If \( 0 \neq [x] \in \text{Ker} \mathcal{A}(a), \) then \([Z] = [J_x] \) is a Jacobi class and satisfies \([Z](0) = [0], \) \([Z](a) = [0] \) but \([Z] \neq [0]. \)

If \( \gamma_v \) has no pair of mutually conjugate points, then for any \( c \neq 0, \) let \( \mathcal{B}_c \) be the solution to (3.4) determined from the boundary data \( \mathcal{B}_c(0) = Id, \) and \( \mathcal{B}_c(c) = 0. \)

Clearly, it satisfies \( \mathcal{B}_c(t)[x] = \bar{P}_v^t [I_x^c(t)] \) where \( I_x^c \) is the unique Jacobi vector field along \( \gamma_v \) such that \( I_x^c(0) = x \) and \( I_x^c(c) = 0. \) A similar argument as in [9] or [6, chapter 5],
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permits us to assert that there exists \( \lim_{c \to +\infty} \frac{dB_c}{dt} \big|_0 = \frac{dB}{dt} \big|_0 \) and taking \( B \) as the solution to (3-4) determined from the initial data \( B(0) = I \) and \( \frac{dB}{dt} \big|_0 \), the continuous dependence of solutions on the initial data gives that \( \lim_{c \to +\infty} B_c(t) = B(t) \) and \( \lim_{c \to +\infty} \frac{dB_c}{dt} \big|_t = \frac{dB}{dt} \big|_t \), with \( B(t) \) non singular for all \( t \geq 0 \).

**Corollary 3.3.** If \( \gamma_v \) has no pair of mutually conjugate points, then for every \( [x] \in \mathcal{N}(v) \), \( [x] \neq [0] \), there exists a Jacobi class \( [I_x] \) such that \( [I_x](t) = \lim_{c \to +\infty} [I_x^c](t) \) and \( \frac{\nabla [I_x^c]}{dt} \big|_t = \lim_{c \to +\infty} \frac{\nabla [I_x]}{dt} \big|_t \). Moreover \( [I_x](t) \neq 0 \) for all \( t \geq 0 \).

**Proof.** It is enough to take \( [I_x](t) = \mathcal{P}_0^t(B(t)[x]) \).

**Remark 3.4.** It should be pointed out that the previous construction is analogous to the one made in [3, chapter 12]. However, the ingredients of equation (3-4) differ from the used ones in this reference.

4. Integral inequalities and their applications

**Theorem 4.1.** Let \((M, g)\) be an \( n(\geq 3) \)-dimensional compact Lorentzian manifold admitting a timelike conformal vector field \( K \). If \((M, g)\) has no conjugate points along its null geodesics, then

\[
\int_{C_K M} \overline{\text{Ric}} \, d\mu_g \leq 0. \tag{4.1}
\]

Moreover, equality holds if and only if \((M, g)\) has constant sectional curvature \( k \leq 0 \).

**Proof.** The inequality (4.1) follows taking into account that if \((M, g)\) has no conjugate points along its null geodesics, then previous inequality (2.2) is valid for any positive real number \( a \).

Now, we will show, in the equality case, that \( \overline{\text{Ric}} = 0 \) on null tangent vectors. For every \( v \in C_K M \), let \( \{E_1, \ldots, E_{n-2}\} \) be a parallel frame along \( \gamma_v \) such that \( \{E_1(0) = e_1, \ldots, E_{n-2}(0) = e_{n-2}\} \) give us an orthonormal basis for \( \langle v \rangle_\perp \cap K_\perp \) with \( \text{Ric}(\gamma_v) = \sum_{j=1}^{n-2} g(R(E_j, \gamma_v)\gamma_v, E_j) \). It is easily seen that \( \{[E_1](t), \ldots, [E_{n-2}](t)\} \) is a basis of \( \mathcal{N}(\gamma_v(t)) \) for all \( t \). We put \( X_i^c(t) = \cos(\pi t/2c)E_i(t) \) for every integer \( c \geq 1 \), and take \( I_i^c \) the unique Jacobi vector field along \( \gamma_v \) such that \( I_i^c(0) = e_i \) and \( I_i^c(c) = 0 \).

It follows that \( \{[I_1^c](t), \ldots, [I_{n-2}^c](t)\} \) is a basis of \( \mathcal{N}(\gamma_v(t)) \) for each \( t \neq c \), since there are no conjugate points along \( \gamma_v \). Therefore, there are smooth functions \( \lambda_{ij}^c \) such that

\[
[X_i^c] = \sum_{j=1}^{n-2} \lambda_{ij}^c[I_j^c], \tag{4.2}
\]

on \( \mathbb{R} - \{c\} \), but taking into account that \( X_i^c(c) = I_i^c(c) = 0 \) it is not hard to show that every function \( \lambda_{ij}^c \) has a smooth extension to \( \mathbb{R} \). From Corollary 3.3, there exist Jacobi classes \( [I_j] = \lim_{c \to +\infty} [I_j^c] \) with \( \{[I_1](t), \ldots, [I_{n-2}](t)\} \) a basis of \( \mathcal{N}(\gamma_v(t)) \) for \( t \geq 0 \), and it is clear that \( \lim_{c \to +\infty} [X_i^c] = [E_i] \). Hence, there are smooth functions \( \lambda_{ij} \) with \( \lambda_{ij}(t) = \lim_{c \to +\infty} \lambda_{ij}^c(t) \) for all \( t \geq 0 \) such that

\[
[E_i] = \sum_{j=1}^{n-2} \lambda_{ij}[I_j] \text{ on } [0, +\infty). \tag{4.3}
\]
We achieve that equality holds in (4·1) and \( \lim_{c \to +\infty} \frac{\nabla_{[X_i]}^c f}{dt} = 0 \) and \( \lim_{c \to +\infty} \frac{\nabla_{[I_j]}^c f}{dt} = \frac{\nabla_{[L]}^c f}{dt} \), a straightforward computation from (4·2) and (4·3) gives \( \lim_{c \to +\infty} d\lambda_{ij}/dt \big|_t = d\lambda_{ij}/dt \big|_t \) for \( t \geq 0 \).

On the other hand, we get

\[
\sum_{i=1}^{n-2} \mathcal{H}_\gamma \left( [X_i^c], [X_i^c] \right) = \int_{-c}^c \left[ \frac{(n-2)\pi^2}{4c^2} \sin^2 \left( \frac{\pi t}{2c} \right) - \cos^2 \left( \frac{\pi t}{2c} \right) \widetilde{\text{Ric}}(\gamma_v') \right] dt \geq 0.
\]

Next, let \( \{f_c\} \) be the following sequence of nonnegative continuous functions

\[
f_c : C_K M \rightarrow \mathbb{R}, \quad v \mapsto \sum_{i=1}^{n-2} \mathcal{H}_\gamma \left( [X_i^c], [X_i^c] \right).
\]

If equality holds in (4·1) and we take into account (2·1) and \( \int_{-c}^c \sin^2(\pi t/2c)dt = c \), then we have

\[
\int_{C_K M} f_c \, d\mu_{\tilde{g}} = \frac{(n-2)\pi^2}{4c} \text{Vol}(C_K M, \tilde{g}).
\]

Therefore, the Fatou lemma implies that \( \int_{C_K M} \liminf f_c \, d\mu_{\tilde{g}} = 0 \), and so \( \liminf f_c(v) = 0 \) for almost all \( v \in C_K M \). Assume now we are considering such a \( v \in C_K M \). Since \( 0 \leq \mathcal{H}_\gamma \left( [X_i^c], [X_i^c] \right) \leq f_c(v) \) holds, we have

\[
\lim \inf \mathcal{H}_\gamma \left( [X_i^c], [X_i^c] \right) = 0,
\]

for every \( i = 1, \ldots, n-2 \).

From Lemma 3·1 we obtain

\[
\mathcal{H}_\gamma \left( [X_i^c], [X_i^c] \right) = \int_{-c}^c \mathcal{I} \left( \sum_{j=1}^{n-2} \frac{d\lambda_{ij}^c}{dt} [I_j], \sum_{j=1}^{n-2} \frac{d\lambda_{ij}^c}{dt} [I_j] \right) dt,
\]

and so, from (4·4), by using the Fatou lemma again, we get

\[
\mathcal{I} \left( \sum_{j=1}^{n-2} \frac{d\lambda_{ij}}{dt} [I_j], \sum_{j=1}^{n-2} \frac{d\lambda_{ij}}{dt} [I_j] \right) = 0, \quad \text{for almost all } t \geq 0.
\]

But \( \mathcal{I} \) is positive definite, and therefore \( \sum_{j=1}^{n-2} \frac{d\lambda_{ij}}{dt} [I_j] = [0] \) on \( [0, +\infty) \) which implies that all \( \lambda_{ij} \) are constant on \( [0, +\infty) \). Therefore, every \( [E_i] \) is a Jacobi class on \( [0, +\infty) \). We achieve that \( \text{Ric}(v) = 0 \) for almost every \( v \in C_K M \). An standard continuity argument gives \( \widetilde{\text{Ric}} = 0 \) on null tangent vectors of \( (M, g) \).

By using [7], we deduce that \( \widetilde{R}(\gamma_v', \gamma_v') = 0 \) for every null vector \( v \) and so [18, proposition 8·28] combined with the Schur theorem gives that \((M, g)\) has constant sectional curvature \( k \).

Since \((M, g)\) must be geodesically complete from a result of completeness by Klinger [15] (this fact can be also deduced from [20]), if \( k > 0 \) then it should be globally isometric to a quotient of the De Sitter space. Nevertheless, it is classical [5] that no discrete subgroup of the isometry group of the De Sitter space which acts properly and discontinuously produces a compact quotient of the De Sitter space. In brief, there exists no compact Lorentzian manifold of positive constant curvature. Therefore, we conclude \( k \leq 0 \) as required.
Lorentzian manifolds with no null conjugate points

Remark 4.2. Compare Theorem 4.1 with [7] where it is assumed that there are no conjugate points along a fixed null geodesic, and the conclusion only concerns that geodesic.

Taking into account [11, proposition 2.3 and lemma 3.4], Theorem 4.1 can be rewritten as follows

**Corollary 4.3.** Let \((M, g)\) be an \(n(\geq 3)\)-dimensional compact Lorentzian manifold admitting a timelike conformal vector field \(K\). If \((M, g)\) has no conjugate points along its null geodesics, then

\[
\int_M \left[ n \widetilde{\text{Ric}}(U) + S \right] h^n d\mu_g \leq 0. \tag{4.7}
\]

Moreover, equality holds if and only if \((M, g)\) has constant sectional curvature \(k \leq 0\).

Corollary 4.3 (or Theorem 4.1) does not remain true for compact Lorentzian surfaces. In fact, let \((M, g)\) be a compact Lorentzian surface admitting a timelike conformal vector field. Now, we recall that there are no null conjugate points on two dimensional Lorentzian manifolds, and that, clearly, \(\text{Ric} = 0\) holds on null tangent vectors in this case. Thus, if Theorem 4.1 were applicable, then \((M, g)\) should have constant Gauss curvature. But now the Gauss–Bonnet theorem for Lorentzian surfaces (see [4], for instance) can be claimed to get that \((M, g)\) must be flat. However, it is known that there exist nonflat Lorentzian tori which admit a timelike Killing vector field [23].

If the vector field \(K\) is assumed, more restrictively, to be Killing, then [11, corollary 3.10] can be improved as follows:

**Theorem 4.4.** Let \((M, g)\) be an \(n(\geq 3)\)-dimensional compact Lorentzian manifold with no conjugate points along its null geodesics and admitting a timelike Killing vector field \(K\), then

\[
\int_M Sh^n d\mu_g \leq 0. \tag{4.8}
\]

Equality holds if and only if \((M, g)\) is isometric to a flat Lorentzian \(n\)-torus up to a (finite) covering. In particular, in this case \(U\) is parallel, the first Betti number of \(M\) is not zero and the Levi–Civita connection of \(g\) is Riemannian.

**Proof.** The inequality is clear from Corollary 4.3 taking into account [11, lemma 3.7]. If equality holds in (4.8), then \(\int_M h^n \text{Ric}(U) d\mu_g = 0\). On the other hand, Corollary 4.3 says that \((M, g)\) has constant sectional curvature \(k \leq 0\); in fact, \(k = 0\) from our assumption. Let us consider the Riemannian metric \(g_R\) defined by \(g_R(X, Y) = g(X, Y) + 2g(X, U)g(Y, U)\), for every \(X, Y \in \mathfrak{X}(M)\). Making use of [11, lemma 3.7], the vector field \(U\) must be parallel. Hence, the Levi–Civita connection associated to \(g_R\) on \(M\) agrees with the Levi–Civita connection of \(g\) (in particular, the \(1\)-form \(\omega\), given by \(\omega(X) = g(X, U)\), is closed and clearly is not exact). Therefore, \((M, g_R)\) is a compact flat Riemannian manifold and then, from [26, corollary 3.4.6], there is a (finite) Riemannian covering \(\rho: (\mathbb{T}^n, g^0_R) \to (M, g_R)\) by a flat Riemannian torus. It is easily seen that \(\rho: (\mathbb{T}^n, g^0) \to (M, g)\) is a Lorentzian covering, where \(g^0_R\) and \(g^0\) are related in a similar way as previous \(g_R\) and \(g\).
Remark 4.5. Kamishima proved in [14, theorem A] that if a compact Lorentzian manifold with constant sectional curvature \( k \in \mathbb{R} \) admits a timelike Killing vector field, then it is geodesically complete and \( k \leq 0 \). Moreover, if \( k = 0 \), then the manifold is affinely diffeomorphic to a Riemannian manifold with non zero first Betti number. So that Corollary 4.4 is a wide extension to this result. It should be noted that Kamishima uses a very different technique in [14] than the present one; in fact his tools are strongly depending on the Lie group machinery of Lorentzian space forms (see [21, 22] for another extension of the Kamishima result).

Theorem 4.4 contains, as a very particular case, the classical Green result in Riemannian geometry ([9], [6, theorem 5.11]) mentioned in the introduction.

Corollary 4.6. Let \((B, g)\) be a compact Riemannian manifold of dimension \( n \geq 2 \) and scalar curvature \( S \). If \((B, g)\) has no conjugate points, then
\[
\int_M S \, d\mu_g \leq 0.
\]
Moreover, equality holds if and only if \((B, g)\) is flat.

Proof. Consider the Lorentzian manifold \((M, g_L) = (S^1 \times B, -g_{can} + g)\), where \( S^1 \) is the unit circle and \( g_{can} \) denote its canonical metric. We may take the vector field \( K \) as the lift to \( S^1 \times B \) of the vector field \( z \mapsto iz \) on \( S^1 \subset \mathbb{C} \). Now, we have only to specialize Theorem 4.4 to this Lorentzian manifold.

Remark 4.7. Theorem 4.4 cannot be extended to the case in which \( K \) is assumed to be conformal. In fact, let \((T^2, g_0)\) be a flat Riemannian torus and let us consider \((S^1, -g_{can})\) as previously. For every \( f \in C^\infty(S^1) \), \( f > 0 \), the Lorentzian manifold \( S^1 \times_f T^2 \) has no conjugate points along its null geodesics [8, theorem 5.4]. If \( U \) is the vector field on \( S^1 \times F \) given by the lift of the vector field \( z \mapsto iz \) on \( S^1 \subset \mathbb{C} \), then \( K = fU \) is a timelike conformal vector field (see [2], for instance).

Making use of the well-known formulae for the curvature of a warped product metric [18, chapter 7], we can write
\[
\int_{S^1 \times T^2} S h^3 d\mu_{gr} = \text{area}(T^2) \int_{S^1} (4f \Delta f + 2 \| \text{grad} f \|_2^2) / f^3 d\mu_{can},
\]
where \( \Delta \), grad and \( \| \| \) are, respectively, the Laplacian, the gradient and the norm of \((S^1, g_{can})\). Now, the classical Green divergence theorem permits us to write
\[
0 = \int_{S^1} \Delta \left( \frac{1}{f} \right) d\mu_{can} = \int_{S^1} \left( -\frac{\Delta f}{f^2} + \frac{2 \| \text{grad} f \|_2^2}{f^3} \right) d\mu_{can}.
\]
Hence, if \( f \) is not constant, then \( \int_{S^1 \times T^2} S h^3 d\mu_{gr} > 0 \).

After showing Theorem 4.4, it might be natural to ask if the function \( h^n \) which appears in the integral inequality (4.8) could be omitted. At this time we do not know the answer, but we think that this is not possible. To support this assertion, note that with our technique the function \( h^n \) appears naturally and plays a non trivial role in the characterization of the equality. On the other hand, whereas in the Riemannian case the absence of conjugate points affects to any geodesic, here we are only making an assumption on null geodesics (in some sense, a set of zero measure within all the geodesics in a Lorentzian manifold).
Lorentzian manifolds with no null conjugate points

As noted above, Corollary 4-3 cannot be used to study Lorentzian surfaces. On the other hand, it does not give consequences from assumptions on timelike conjugate points. However, by means of a trick, the following result can be obtained.

**Corollary 4-8.** Let \( (M, g) \) be a compact Lorentzian surface admitting a timelike Killing vector field \( K \). If \( (M, g) \) has no conjugate points along its timelike geodesics, then \( (M, g) \) must be flat.

**Proof.** Because \( M \) is 2-dimensional, none of its null geodesics has conjugate points. Using this fact and the form of the geodesics in a semi-Riemannian product, we conclude that the Lorentzian manifold \( (M \times S^1, g + g_{can}) \) has no conjugate points along its null geodesics. Moreover, \( K \) naturally induces a timelike Killing vector field on this semi-Riemannian product. From Theorem 4-4 we obtain

\[
\int_M G h^3 d\mu_g \leq 0,
\]

where \( G = (1/2)S \) is the Gauss curvature of \( M \). Moreover equality holds if and only if \( (M, g) \) is flat. On the other hand, the integrand in (4-7) is \( [3\text{Ric}(U) + S]h^3 = [-3G + 2G]h^3 = -Gh^3 \). So, Corollary 4-3 can also be claimed to get the opposite inequality to the previous one.

5. **Examples**

1. **Compact standard static** Lorentzian manifolds are warped products \( B \times_f F \) with \( (B, g_B) \) a compact Riemannian manifold, \( \dim B = n \geq 2 \), and \( (F, g_F) = (S^1, -g_{can}) \). If \( K \) is the timelike vector field on \( B \times S^1 \) given by the lift of the vector field \( z \mapsto iz \) on \( S^1 \subset \mathbb{C} \), then \( h = 1/\sqrt{-g_f(K, K)} = 1/(f \circ \pi_B) \). Moreover, from [18, lemma 12-37] \( K \) is Killing.

Assume \( B \times_f S^1 \) has no conjugate points along its null geodesics. Using [18, chapter 7], Corollary 4-3 gives

\[
\int_B \left[ \frac{(n - 1)\Delta f}{f^{n+1}} + \frac{S^B}{f^n} \right] d\mu_{g_B} \leq 0,
\]

where \( \Delta \) and \( S^B \) are the Laplacian and the scalar curvature of \( (B, g_B) \), respectively. Taking now into account \( \Delta(\frac{1}{nf^n}) = \frac{\Delta f}{f^{n+1}} - \frac{n+1}{f^{n+2}} \|\text{grad} f\|^2 \), the classical Green divergence theorem permits us to write

\[
\int_B \frac{\Delta f}{f^{n+1}} d\mu_{g_B} = (n + 1) \int_B \frac{\|\text{grad} f\|^2}{f^{n+2}} d\mu_{g_B} \geq 0.
\]

Therefore, if \( B \times_f S^1 \) has no conjugate points along its null geodesics, \( \int_B \frac{S^n}{f^n} d\mu_{g_B} \leq 0 \), and equality holds if and only if \( f \) is constant and \( (B, g_B) \) is flat.

2. **Generalized Robertson–Walker compact spacetimes** are warped products \( B \times_f F \) with \( (B, g_B) = (S^1, -g_{can}) \) and \( (F, g_F) \) a compact Riemannian manifold with \( \dim F = n \geq 2 \). If \( U \) is the vector field on \( S^1 \times F \) given by the lift of the vector field \( z \mapsto iz \) on \( S^1 \subset \mathbb{C} \), then \( K = fU \) is a timelike conformal vector field, [2] and \( h = 1/(f \circ \pi_{S^1}) \). Assume \( S^1 \times_f F \) has no conjugate points along its null geodesics (this happens if the fiber \( (F, g_F) \) has no conjugate points [8, theorem 5-4]). A straightforward computation, by using the curvature formula of a warped product [18, chapter 7],
permits us to write (4·7) as follows
\[ n(n-1)\text{Vol}(F)\int_{S^1} \left[ \frac{(f')^2}{f^3} - \frac{f''}{f^2} \right] d\mu_{\text{g}_\text{can}} + \int_F S^F d\mu_{\text{g}_\text{rr}} \int_{S^1} \frac{1}{f^3} d\mu_{\text{g}_\text{can}} \leq 0, \tag{5·1} \]
where \( f' \) denotes \( U(f) \). Observe that \( \Delta(-1/f) = -f''/f^2 + 2(f')^2/f^3 \) (\( \Delta \) denotes here the Laplacian operator for \(-g_{\text{can}}\); i.e. \( \Delta = -\Delta^0 \), where \( \Delta^0 = d^2/d\theta^2 \) is the usual Laplacian of \( S^1 \)). Now, with the help of the classical Green divergence theorem, (5·1) is rewritten as follows
\[ \int_F S^F d\mu_{\text{g}_\text{rr}} \int_{S^1} \frac{1}{f^3} d\mu_{\text{g}_\text{can}} \leq n(n-1)\text{Vol}(F)\int_{S^1} (f')^2/f^3 d\mu_{\text{g}_\text{can}}. \tag{5·2} \]
In the equality case, \( S^1 \times_f F \) has constant sectional curvature \( k \leq 0 \). Therefore \( \text{Ric}(U) = -nk = -nf''/f \). On the other hand, \( \Delta(\ln f) = -f''/f + (f')^2/f^2 \), and so we use again the divergence theorem to get that \( f \) is constant (i.e. \( S^1 \times_f F \) is, in fact, a semi-Riemannian product) and \( k = 0 \). Therefore the fiber \((F, g_F)\) must be flat.

REFERENCES
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